Numbers – Complex

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Complex Numbers are two (2) part numbers:

$$\begin{aligned} z &= Part\ \#1 + i\ Part\ \#2 \\ &= a + ib = (a,b) \end{aligned}$$

where a & b are real numbers and "i" is called the imaginary unit – lousy name.

Imaginary Unit: $i = \sqrt{-1}$

Key: Positive Integer "I" can be written as $I = 4Q + R$ where Q is the Quotient and R is the Remainder satisfying $0 \le R \le 3$. Hence,

$$i^I = i^{4Q+R} = (i^4)^Q * i^R = 1^Q * i^R = \begin{cases} 1\ ;\ R = 0 \\ \text{i}\ ;\ R = 1 \\ \text{-1}\ ;\ R = 2 \\ \text{-i}\ ;\ R = 3 \end{cases}$$

We have the following:

$i = i^1 = \sqrt{-1}$ Definition

$i^2 = -1$

$i^3 = i^2 * i^1 = -1 * i = -i$

$i^4 = i^2 * i^2 = (-1) * (-1) = 1$

$i^5 = i^4 * i^1 = i$

$i^6 = i^4 * i^2 = -1$

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(1) **Question**: $\mathbf{i}^{79} = ?$

Solution:

Step	Equation	Reason
0	$\mathbf{i}^{79} =$	
1	$\mathbf{i}^{4*19+3} =$	
2	$\left(\mathbf{i}^{4}\right)^{19}\mathbf{i}^{3} =$	
3	$\left(1\right)^{19}\mathbf{i}^{3} =$	
4	$\mathbf{i}^{3} =$	
5	$-\mathbf{i}$	

(2) **Question**: $\frac{1}{\mathbf{i}^{19}} = ?$

Solution:

Step	Equation	Reason
0	$\frac{1}{\mathbf{i}^{19}} =$	
1	$\frac{1}{\mathbf{i}^{4*4+3}} =$	
2	$\frac{1}{\left(\mathbf{i}^{4}\right)^{4}\mathbf{i}^{3}} =$	
3	$\frac{1}{\left(1\right)^{4}\mathbf{i}^{3}} =$	
4	$\frac{1}{\mathbf{i}^{3}} =$	
5	$\frac{1}{-\mathbf{i}} =$	
6	$-\frac{1}{\mathbf{i}}*\frac{\mathbf{i}}{\mathbf{i}} =$	
7	$-\frac{\mathbf{i}}{\mathbf{i}^{2}} =$	
8	$-\frac{\mathbf{i}}{-1} =$	
9	$\mathbf{i}$	

(3) **Question**: $\mathbf{i}^{-173} = ?$
Solution:

Step	Equation	Reason
0	$\mathbf{i}^{-173} =$	
1	$\frac{1}{\mathbf{i}^{173}} =$	
2	$\frac{1}{\mathbf{i}^{4*43+1}} =$	
3	$\frac{1}{\left(\mathbf{i}^{4}\right)^{4}\mathbf{i}^{1}} =$	
4	$\frac{1}{(1)^{4}\mathbf{i}^{1}} =$	
5	$\frac{1}{\mathbf{i}} =$	
6	$\frac{1}{\mathbf{i}} * \frac{\mathbf{i}}{\mathbf{i}} =$	
7	$\frac{\mathbf{i}}{\mathbf{i}^{2}} =$	
8	$\frac{\mathbf{i}}{-1} =$	
9	$-\mathbf{i}$	

Definitions: A **complex number** has the "standard" form

$$z = a + ib = (a, b) \text{ where } i = \sqrt{-1}$$
$$= (\text{Real Part}) + i(\text{Imaginary Part}) = (\text{Real Part}, \text{Imaginary Part})$$

so

$$"a" = \text{Real Part of } z$$
$$"b" = \text{Imaginary Part of z}$$

Also

$$\bar{z} = a - ib = \text{conjugate of z} \quad [\text{Change the SIGN of the Imag Part}]$$

&

$$|z| = \sqrt{a^2 + b^2} = \text{magnitude of z}$$

Note: Every real number "x" can be written as a complex number: $x = x + 0i$

(4) **Question**: Find the conjugate of $\mathbf{z}_1 = 2 - 3\mathbf{i}$

Solution:

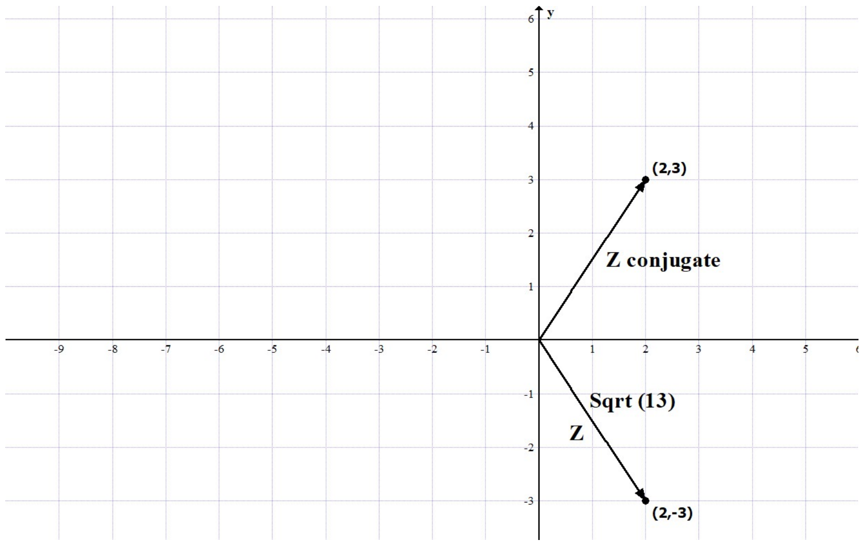
The conjugate of $\mathbf{a} + \mathbf{i}\,\mathbf{b}$ is $\mathbf{a} - \mathbf{i}\,\mathbf{b} = 2 + 3\mathbf{i}$.

(5) **Question**: Find the magnitude of $\mathbf{z}_1 = 2 - 3\mathbf{i}$.

Solution:

The magnitude of $\mathbf{a} + \mathbf{i}\,\mathbf{b}$ is $\sqrt{\mathbf{a}^2 + \mathbf{b}^2} = \sqrt{(2)^2 + (3)^2} = \sqrt{13}$.

Since a complex number $\mathbf{z} = \mathbf{a} + \mathbf{ib}$ can be considered as the ordered pair $(2, -3)$, we can graph it using the Number Plane:



Note: We can think of a complex number as a **vector** since it has a magnitude and direction.

Operations with Complex Numbers:

Let

$$z_1 = a + ib = 2 + 3i$$
$$z_2 = c + id = -1 + 5i$$

Sum: $z_1 + z_2 = (a + ib) + (c + id) = \overbrace{(a + c)}^{\text{Real Part}} + \overbrace{(b + d)}^{\text{Imag Part}} i$

Example: $z_1 + z_2 = (2 + 3i) + (-1 + 5i) = 1 + 8i$

Difference: $z_1 - z_2 = (a + ib) - (c + id) = \overbrace{(a - c)}^{\text{Real Part}} + \overbrace{(b - d)}^{\text{Imag Part}} i$

Example: $z_1 - z_2 = (2 + 3i) - (-1 + 5i) = 3 - 2i$

Product: $z_1 * z_2 = (ac - bd) + (ad + bc)i$

Example: $z_1 * z_2 = (2 + 3i) * (-1 + 5i) = -2 + 10i - 3i + 15i^2 = -2 + 7i - 15 = -17 + 7i$

Quotient: $\frac{z_1}{z_2} = \frac{a + ib}{c + id} = \frac{a + ib}{c + id} * \frac{c - id}{c - id} = \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2}$

Example: $\frac{z_1}{z_2} = \frac{2+3i}{-1+5i} = \frac{2+3i}{-1+5i} * \frac{-1-5i}{-1-5i} = \frac{(2+3i)*(-1-5i)}{(-1+5i)*(-1-5i)} = \frac{-2-10i-3i-15i^2}{1+5i-5i-25i^2}$

$= \frac{13-13i}{26} = \frac{1}{2} - \frac{1}{2}i$

Below are a few more examples in multiple choice format:

In **Examples 6-9,** define

$\mathbf{z}_1 = 2 - 3\mathbf{i}$
$\mathbf{z}_2 = 5 + 4\mathbf{i}$

(6) **Typical Question**: The real part of $\mathbf{z}_1 + \mathbf{z}_2$ equals

A) -7
B) -1
C) 1
D) 7

ANSWER: D

Solution:

Step	Equation	Reason
0	$\mathbf{z}_1 + \mathbf{z}_2 =$	
1	$(2-3\mathbf{i})+(5+4\mathbf{i}) =$	
2	$(2+5)+(-3+4)\mathbf{i} =$	
3	$7+\mathbf{i}$	

(7) **Typical Question**: The imaginary part of $\mathbf{z}_1 - \mathbf{z}_2$ equals

A) -7
B) -3
C) 3
D) 7

ANSWER: A

Solution:

Step	Equation	Reason
0	$\mathbf{z}_1 - \mathbf{z}_2 =$	
1	$(2-3\mathbf{i})-(5+4\mathbf{i})=$	
2	$(2-5)+(-3-4)\mathbf{i}=$	
3	$-3-7\mathbf{i}$	

(8) **Typical Question**: The product $\mathbf{z}_1 * \mathbf{z}_2$ equals

A) $-22-7\mathbf{i}$
B) $-22+7\mathbf{i}$
C) $22-7\mathbf{i}$
D) $22+7\mathbf{i}$

ANSWER: C

Solution:

Step	Equation	Reason
0	$\mathbf{z}_1 * \mathbf{z}_2 =$	
1	$(2-3\mathbf{i})*(5+4\mathbf{i})=$	
2	$10+8\mathbf{i}-15\mathbf{i}-12\mathbf{i}^2 =$	
3	$10+8\mathbf{i}-15\mathbf{i}-12(-1)=$	
4	$10+8\mathbf{i}-15\mathbf{i}+12=$	
5	$22-7\mathbf{i}$	

(9) **Typical Question**: The quotient $\frac{\mathbf{z}_1}{\mathbf{z}_2}$ equals

A) $\frac{-2-23\mathbf{i}}{41}$

B) $\frac{-2+23\mathbf{i}}{41}$

C) $\frac{-2+23\mathbf{i}}{41}$

D) $\frac{2+23\mathbf{i}}{41}$

ANSWER: A

Solution:

Step	Equation	Reason
0	$\frac{\mathbf{z}_1}{\mathbf{z}_2} =$	
1	$\frac{2-3\mathbf{i}}{5+4\mathbf{i}} =$	
2	$\frac{2-3\mathbf{i}}{5+4\mathbf{i}} * \frac{5-4\mathbf{i}}{5-4\mathbf{i}} =$	
3	$\frac{10-8\mathbf{i}-15\mathbf{i}+12\mathbf{i}^2}{25-16\mathbf{i}^2} =$	
4	$\frac{10-8\mathbf{i}-15\mathbf{i}+12(-1)}{25-16(-1)} =$	
5	$\frac{10-8\mathbf{i}-15\mathbf{i}-12}{25+16} =$	
6	$\frac{-2-23\mathbf{i}}{41} =$	

