FUNctions: Continuity

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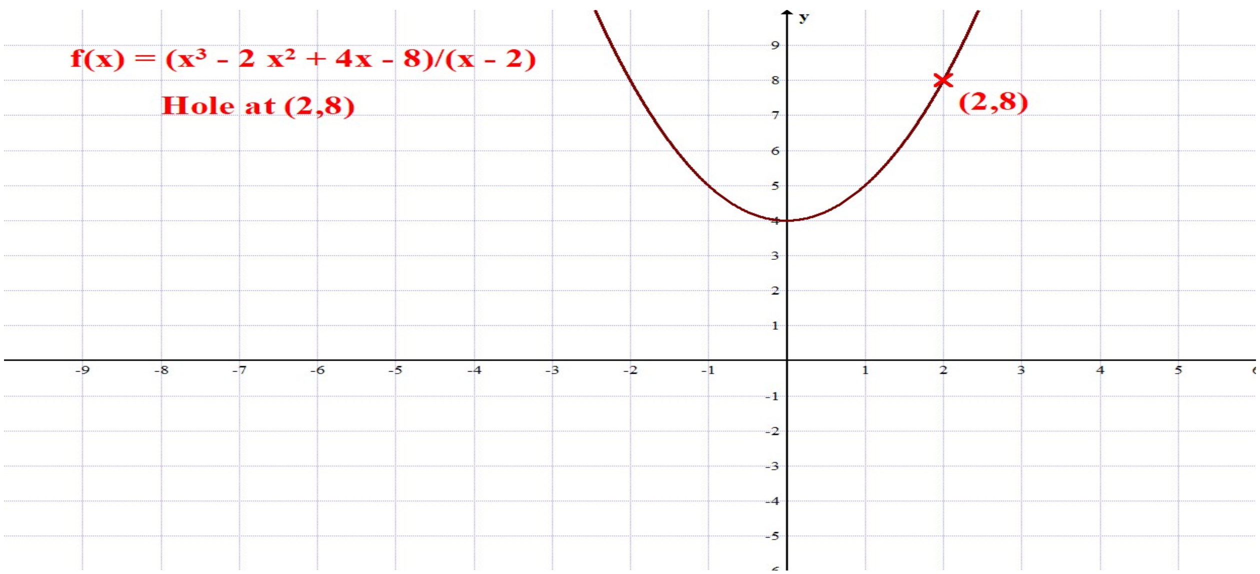
If a function does NOT have *any* breaks in its graph, it is said to be a **continuous function**. There are three (3) types of breaks (called discontinuities) our graphs may have:

1. Hole: $\mathbf{y} = \mathbf{f}(\mathbf{x}) = \dfrac{\mathbf{x}^3 - 2\mathbf{x}^2 + 4\mathbf{x} - 8}{\mathbf{x} - 2}$

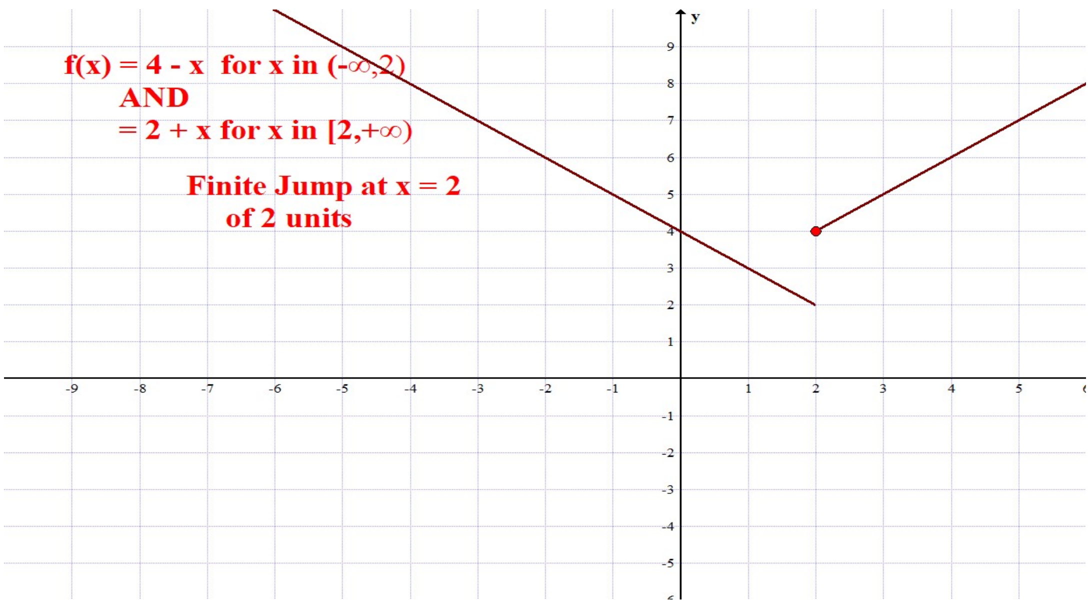
The $\textbf{Dom f} = \mathbb{R}_{\mathbf{x}} \setminus \{2\}$ and we denote the hole in the graph at (2,8) by an "x"



Note: This analysis just gives us a small portion of the graph although the entire graph was drawn.

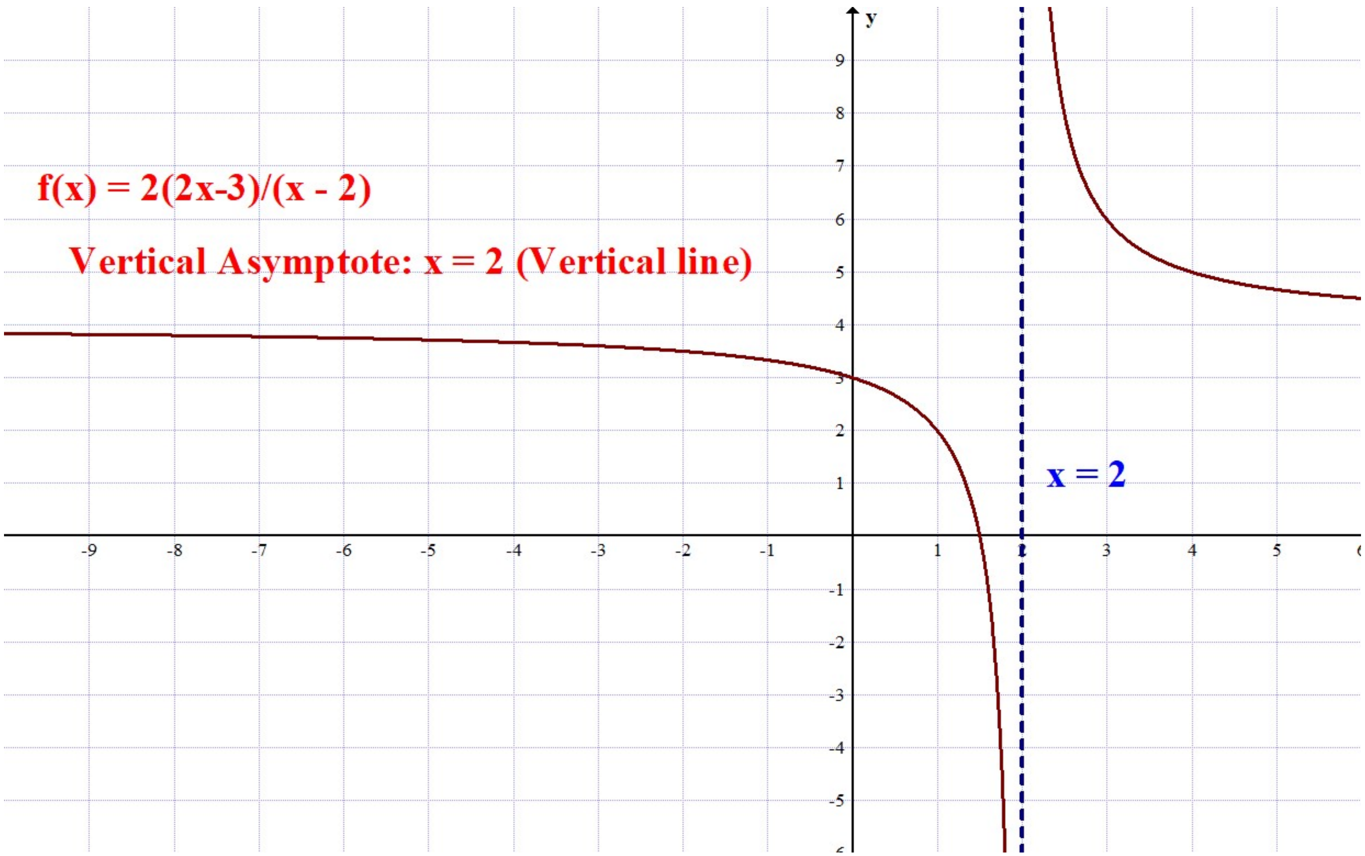
2. Finite Jump: $\mathbf{y} = \mathbf{f}(\mathbf{x}) = \begin{cases} 4 - \mathbf{x} & \text{if } \mathbf{x} \in (-\infty, 2) \\ 2 + \mathbf{x} & \text{if } \mathbf{x} \in [2, +\infty) \end{cases}$

   The graph has a finite jump of 2 units at x = 2.



3. Vertical asymptote: $\mathbf{f}(\mathbf{x}) = \frac{2(2\mathbf{x}-3)}{\mathbf{x}-2}$

The vertical line x = 2 is called a **vertical asymptote** of f. Note that as the x-values get "closer and closer" to x = 2, the corresponding f(x) values get bigger without bound (we say "goes to positive infinity) on one side of x = 2 and smaller without bound (we say "goes to negative infinity) on the other.



Theorem (Fundamental Theorem of Continuity): Let $\mathbf{f}$ be a function defined on an interval $\mathbf{I}$. Assume

1. $\mathbf{f}$ is continuous for all $\mathbf{x} \in \mathbf{I}$ (NO Breaks)
2. $\mathbf{f}(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in \mathbf{I}$ (NO x-intercept POINTS)

Then either

- $\mathbf{f}(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathbf{I}$

or

- $\mathbf{f}(\mathbf{x}) < 0$ for all $\mathbf{x} \in \mathbf{I}$

Note:

1. **Pos f** will represent the x-axis regions where $\mathbf{f}(\mathbf{x}) > 0$
2. **Neg f** will represent the x-axis regions where $\mathbf{f}(\mathbf{x}) < 0$

Example 01: Given $\mathbf{f}(\mathbf{x}) = \mathbf{x}^3 - 3\mathbf{x}$, find where it is negative, zero, and positive.

Solution:

We first find the x-intercept points:

$$\mathbf{f}(\mathbf{x}) = \mathbf{x}^3 - 3\mathbf{x} = \mathbf{x}\left(\mathbf{x}^2 - 3\right) \stackrel{\textbf{SET}}{=} 0 \Rightarrow \mathbf{x} = 0, \pm\sqrt{3} \Rightarrow \left(-\sqrt{3}, 0\right), (0,0), \left(\sqrt{3}, 0\right)$$

These three (3) points divide the x-axis into four intervals:

$$\left(-\infty, -\sqrt{3}\right), \left(-\sqrt{3}, 0\right), \left(0, \sqrt{3}\right), \left(\sqrt{3}, +\infty\right)$$

Each of these intervals satisfies the hypothesis in the above theorem so we can select a representative point in each interval to determine the sign in the entire interval:

$$\underbrace{\overbrace{\qquad\qquad\qquad\qquad}^{-}}_{\mathbf{f}(-2)<0}\ -\sqrt{3}\ \underbrace{\overbrace{\qquad\qquad}^{+}}_{\mathbf{f}(-1)>0}\ 0\ \underbrace{\overbrace{\qquad\qquad}^{-}}_{\mathbf{f}(1)<0}\ \sqrt{3}\ \underbrace{\overbrace{\qquad\qquad\qquad\qquad}^{+}}_{\mathbf{f}(2)>0}$$

Therefore, $\textbf{Pos f} = \left(-\sqrt{3},0\right)\cup\left(\sqrt{3},+\infty\right)$ and $\textbf{Neg f} = \left(-\infty,-\sqrt{3}\right)\cup\left(0,\sqrt{3}\right)$

The function is positive in the *green* regions and negative in the *red* regions: