FUNctions: Behavior at/toward Infinity

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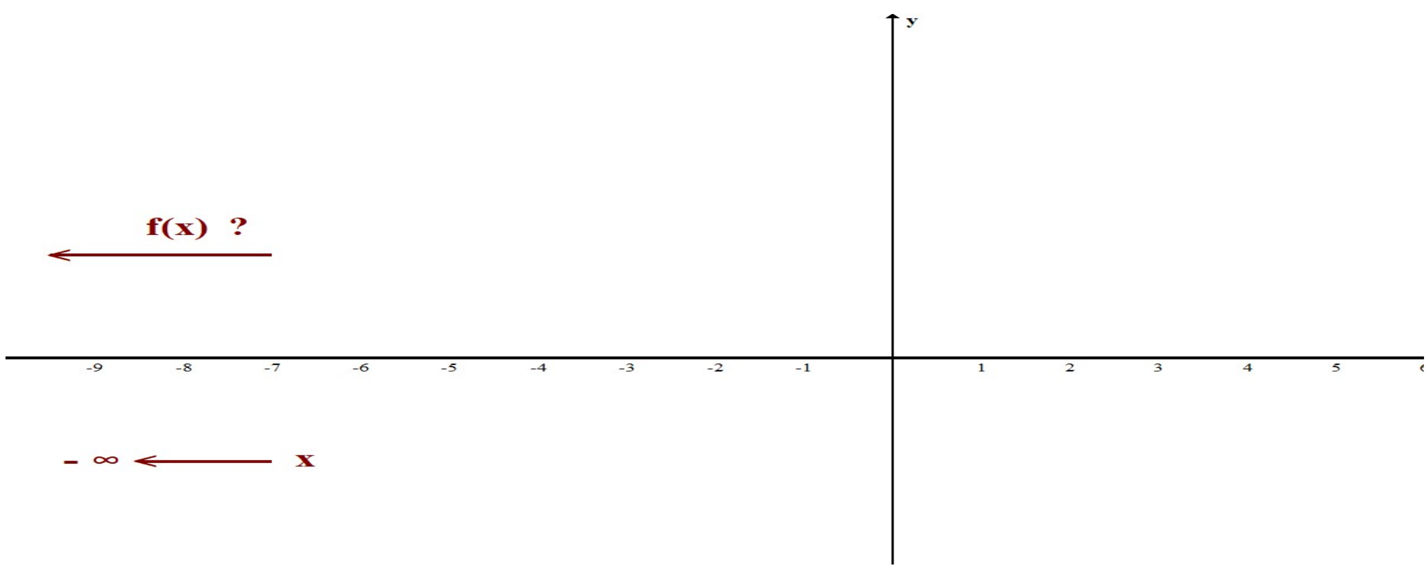
For the sake of discussion, let us assume that the domain of the function f under consideration is all the real numbers: $\mathbb{R}_x$. To get a rough approximation to the graph of f, we plot some points on its graph including the intercept points and then connect them like we did with our "dot-to-dot" colorings book when we were young assuming that there are no breaks in the graph. However, since we can only plot a finite number of points, our graphical representation will be lacking. To improve our representation, we determine if the f(x) values have a pattern as the x-values increase (decrease) without bound:



In symbols, the question is $|\mathbf{x}| \to +\infty \Rightarrow \mathbf{f}(\mathbf{x}) \to ?$

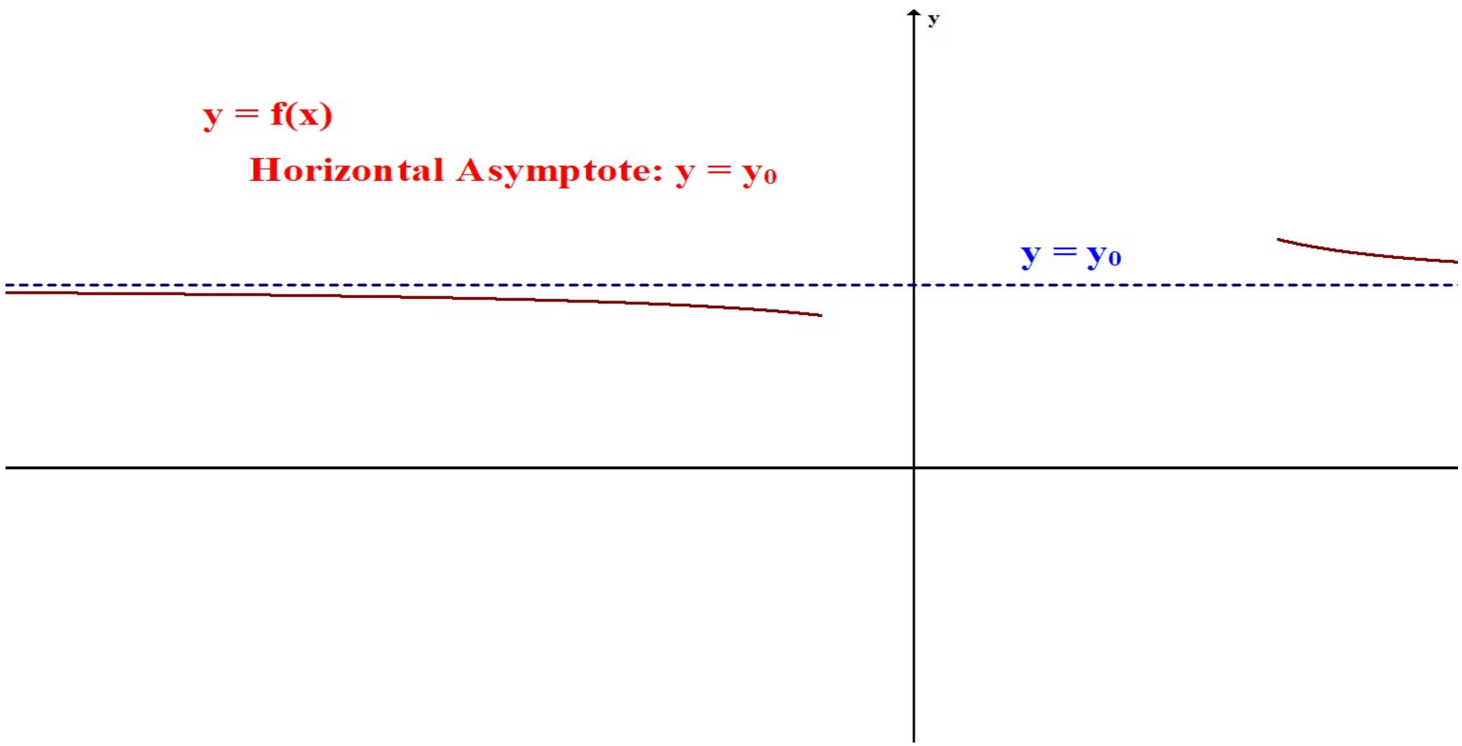
In *College Algebra*, we concentrate on two (2) patterns the f(x) values can have:

1. $|\mathbf{x}| \to +\infty \Rightarrow \mathbf{f}(\mathbf{x}) \to \pm\infty$
   As the x-values increase (decrease) without bound, the corresponding f(x) values increase (decrease) without bound.
2. $|\mathbf{x}| \to +\infty \Rightarrow \mathbf{f}(\mathbf{x}) \to \mathbf{y}_0 \in \mathbb{R}_{\mathbf{y}}$
   As the x-values increase (decrease) without bound, the corresponding f(x) values approach a number $\mathbf{y}_0$.

In the second case, we obtain what we call a *horizontal asymptote* of f: $\mathbf{y} = \mathbf{y}_0$

Definition: A horizontal line $\mathbf{y} = \mathbf{y}_0$ is a **horizonal asymptote** of f if as the x-values increase (decrease) without bound (we say "goes to $+\infty$ (or $-\infty$)") , the corresponding (x, f(x)) points get "closer and closer" to the line $\mathbf{y} = \mathbf{y}_0$.

Note: In other words, the graph of f gets "closer and closer" to $\mathbf{y} = \mathbf{y}_0$ as the x-values approach $+\infty$ (or $-\infty$).



Note: A horizontal asymptote $\mathbf{y} = \mathbf{y}_0$ may or may not intercept the graph.

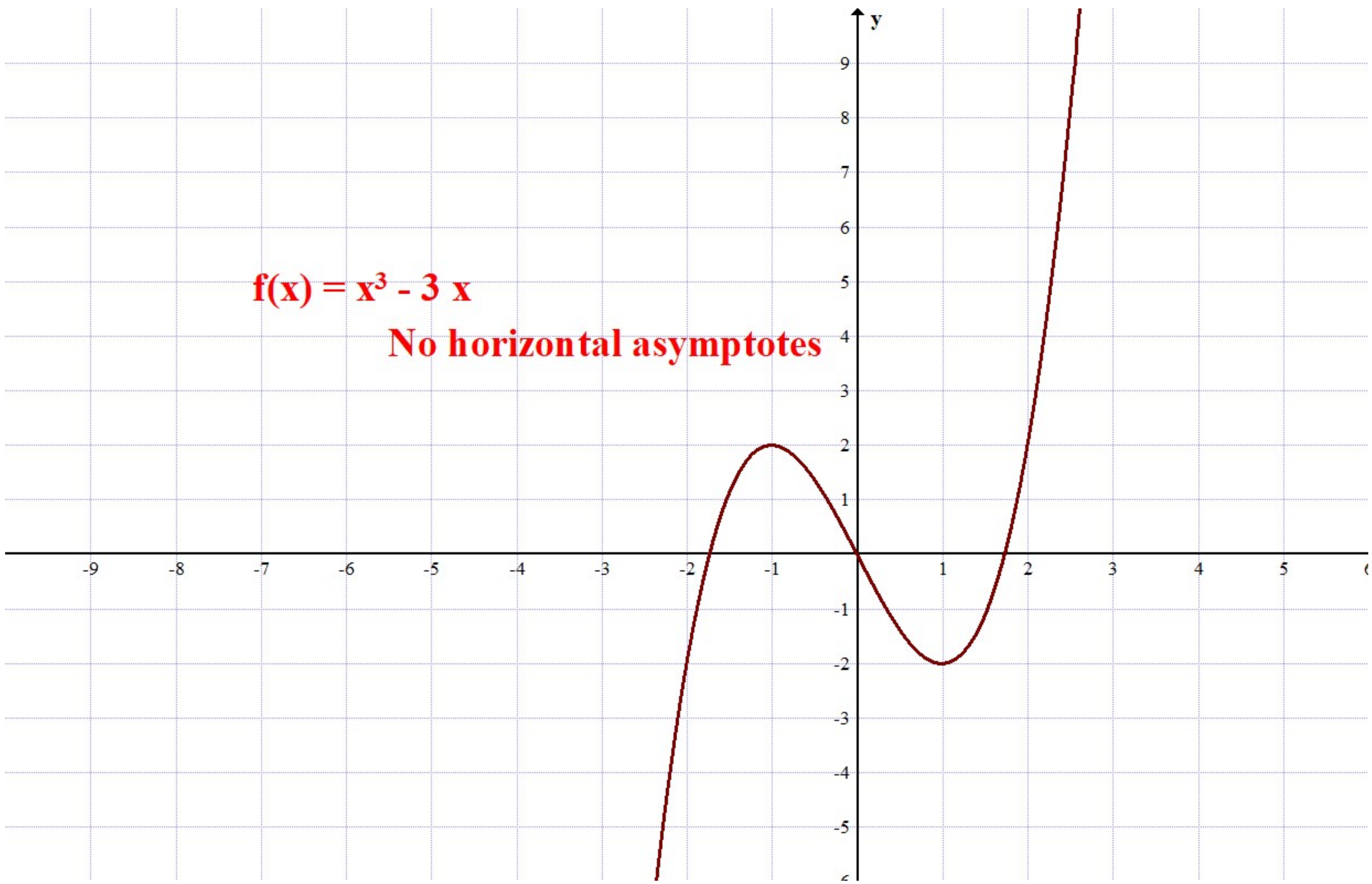
Note: There can be 0, 1,or 2 horizontal asymptotes.

In the examples below, we just want to identify the behavior as $|\mathbf{x}| \to +\infty$, given the graph.

Example 01: $y = f(x) = x^3 - 3x$

Analysis:

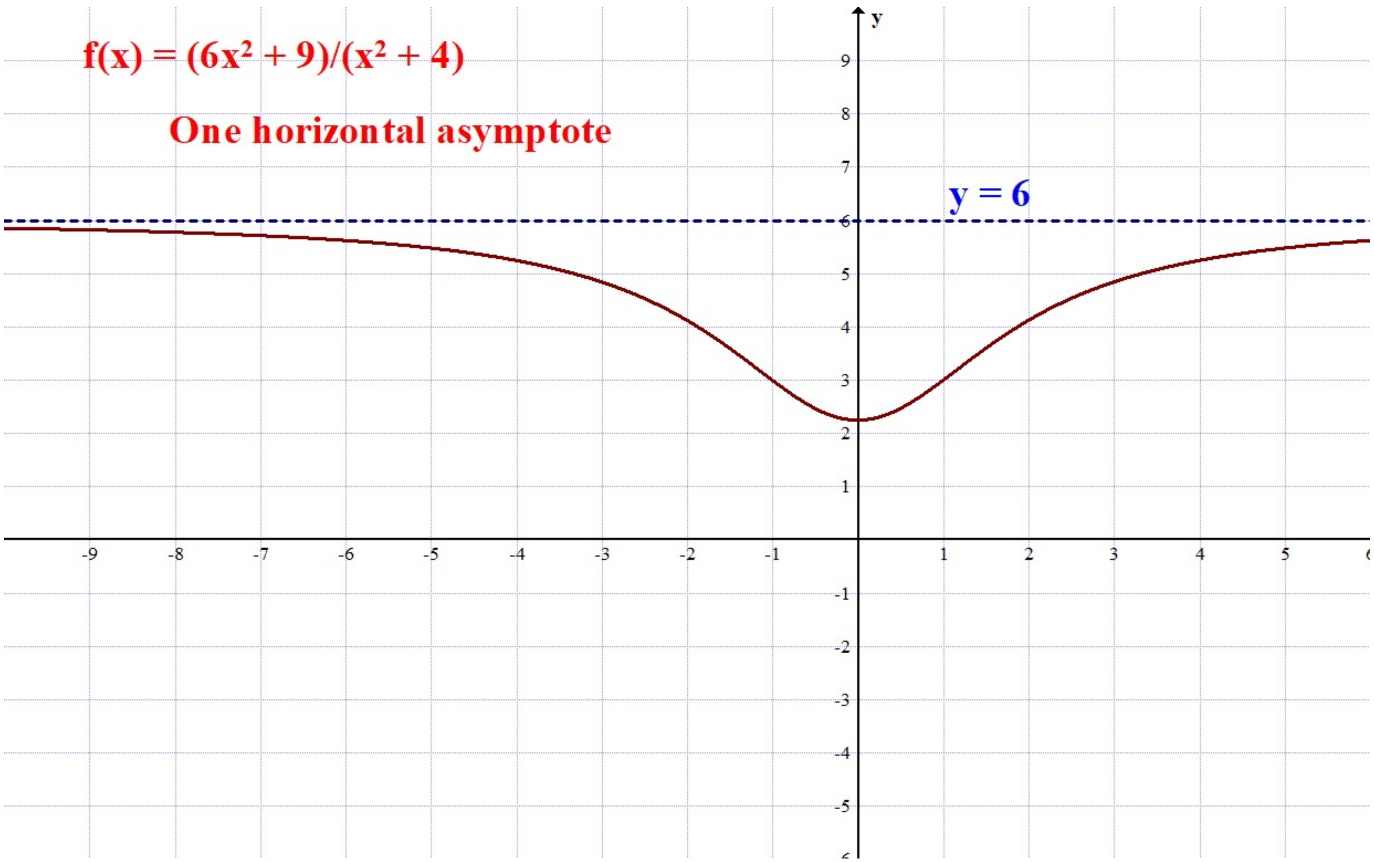
Considering the graph of f we see . Therefore there are no horizontal asymptotes.



Example 02: $y = f(x) = \dfrac{6x^2 + 9}{x^2 + 4}$

Analysis:

Considering the graph of f, we see $x \to -\infty \Rightarrow f(x) \to 6$; $x \to +\infty \Rightarrow f(x) \to 6$. Therefore y = 6 is a horizontal asymptote of f.



Example 03: $f(x) = \frac{\sqrt{2+4x^2}}{x}$

Analysis:

Considering the graph of f, we see $x \to -\infty \Rightarrow f(x) \to -2$; $x \to +\infty \Rightarrow f(x) \to +2$. Therefore y = -2 **AND** y=2 are horizontal asymptotes of f.