FUNctions – Range

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Basic Function Idea/Concept – For each "allowable input:", there corresponds a "unique output:

Allowable Input $\overset{\textbf{implies}}{\Rightarrow}$ **Unique Output**

The functions we consider involve just real numbers and are referred to as "real-valued functions":

Assumption: We consider real-valued functions, that is, functions that take real #'s to real #'s

Recall from our set theory material:

1. The symbol "$\subseteq$" means "contained in" or "a subset of"
2. The symbol "$\in$" means "is an element of " or "member of"
3. $\mathbb{R}_x$ represents a copy of the *horizontal* number line
4. $\mathbb{R}_y$ represents a copy of the *vertical* number line

Using these symbols, we present the technical definition of a function:

Definition: A **function** $\mathbf{f}$ from a set $\mathbf{X} \subseteq \mathbb{R}_x$ unto a set $\mathbf{Y} \subseteq \mathbb{R}_y$ is a correspondence that associates with each $\mathbf{x} \in \mathbf{X}$ one and only one $\mathbf{y} \in \mathbf{Y}$. This is sometimes denoted $\mathbf{f : X \to Y}$. Also, the correspondence, rule, formula, … is frequently denoted by **f(x): y = f(x).**

$\mathbf{X} \subseteq \mathbb{R}_x$ is called the **domain**: **Domain f = Dom f**

Note: The *domain* of $\mathbf{f}$ is a subset of all the real numbers on the horizontal number line.

$\mathbf{Y} \subseteq \mathbb{R}_y$ is called the **range**: $\mathbf{Range\ f} = \mathbf{Rng\,f}$

Note: The *range* of $\mathbf{f}$ is a subset of all the real numbers on the vertical number line.

To *officially* specify a function, we need

1. Name
2. Symbol
3. Domain
4. Correspondence (rule, formula, …)
5. Range

Note, however, that all of this information is usually not initially given and but can be determined. In particular, the **range** of **f** is seldom given.

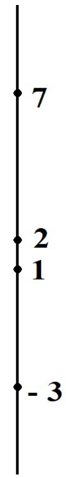
The **range** of $\mathbf{f}$ may be defined two (2) ways:

1. **Explicitly** – the range is given

(1) Consider $\mathbf{f}$ defined by the following table:

x	**y = f(x)**
3	**7**
-2	**1**
4	**-3**
6	**2**

Its range is the *vertical* "$\mathbf{y}$" column: $\mathbf{Range\ f} = \{-3, 1, 2, 7\}$. Graphically,



2. **Implicitly** – the range is defined by

$$\textbf{Range f} = \{\mathbf{y} = \mathbf{f}(\mathbf{x}) \mid \mathbf{x} \in \textbf{Dom f}\}$$

that is, the set of all y values we obtain by calculating y = **f**(x) for *all* values in the domain.

If we know the graph of $\mathbf{f}$, its range is the projection of this graph onto the y-axis. If we do NOT know the graph of $\mathbf{f}$ **AND** we can solve $\mathbf{y} = \mathbf{f}(\mathbf{x})$ for $\mathbf{x}$

$$\mathbf{x} = \text{"}\mathbf{y}\text{" stuff}$$

we can see if there are any "$\mathbf{y}$" values that must be excluded from the maximum range which is the set of all vertical real numbers: $\mathbb{R}_y$ There are two (2) ways of excluding a real number $\mathbf{y}$ from being in the range of a function $\mathbf{f}$:

1. **Division by zero**: A value for $\mathbf{y}$ can NEVER make the denominator zero!

$$\frac{\text{Numerator}}{\text{Denominator} \neq 0}$$

2. **Negative under an even root**: A value for $\mathbf{y}$ can NEVER make an expression under an even root negative!

$$\sqrt[\text{EVEN}]{\text{Expression} \not< 0}$$

Note: These restrictions are the same as for the domain **except** they now apply to $\mathbf{y}$ values.

(2) Find the range of $\mathbf{f}(\mathbf{x}) = 4$.

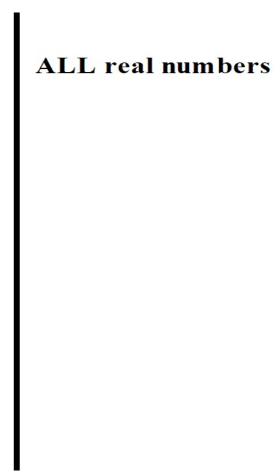
Obviously, the range contains only one element: $\textbf{Range f} = \{4\}$:

(3) Find the range of $\mathbf{f}(\mathbf{x}) = 4\mathbf{x} - 9$.

From our prior experience with straight lines, we know that the range is **Range f** $= \mathbb{R}_y$. To see this another way, we can solve $\mathbf{y} = \mathbf{f}(\mathbf{x}) = 4\mathbf{x} - 9$ for $\mathbf{x}$:

Step	Equation	Reason
0	$\mathbf{y} = \mathbf{f}(\mathbf{x}) = 4\mathbf{x} - 9$	
1	$\mathbf{y} = 4\mathbf{x} - 9$	
2	$\mathbf{y} + 9 = 4\mathbf{x}$	
3	$\frac{\mathbf{y}+9}{4} = \mathbf{x}$ OR $\mathbf{x} = \frac{\mathbf{y}+9}{4}$	

Since $\mathbf{x} = \frac{\mathbf{y}+9}{4}$ does not contain any restrictions on $\mathbf{y}$, we also see **Range f** $= \mathbb{R}_y$:



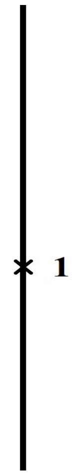
(4) Find the range of $\mathbf{f}(\mathbf{x}) = \frac{\mathbf{x}+3}{\mathbf{x}}$.

Since it is NOT obvious what the graph of **f** is, we solve $\mathbf{y} = \mathbf{f}(\mathbf{x}) = \frac{\mathbf{x}+3}{\mathbf{x}}$ for $\mathbf{x}$:

Step	Equation	Reason
0	$\mathbf{y} = \mathbf{f}(\mathbf{x}) = \frac{\mathbf{x}+3}{\mathbf{x}}$	
1	$\mathbf{y} = \frac{\mathbf{x}+3}{\mathbf{x}}$	
2	$\mathbf{xy} = \mathbf{x}+3$	
3	$\mathbf{xy} - \mathbf{x} = 3$	
4	$\mathbf{x}(\mathbf{y}-1) = 3$	
5	$\mathbf{x} = \frac{3}{\mathbf{y}-1}$	

Since $\mathbf{x} = \frac{3}{\mathbf{y}-1}$, we must exclude $\mathbf{y} = 1$ since it makes the denominator zero:

Range f $= \mathbb{R}_{\mathbf{y}} \setminus \{1\} = (-\infty, 1) \cup (1, +\infty)_{\mathbf{y}}$:



(5) Find the range of $\mathbf{f}(\mathbf{x}) = \dfrac{\mathbf{x}-1}{4\mathbf{x}+1}$.

Since it is NOT obvious what the graph of $\mathbf{f}$ is, we solve $\mathbf{y} = \mathbf{f}(\mathbf{x}) = \dfrac{\mathbf{x}-1}{4\mathbf{x}+1}$ for $\mathbf{x}$:

Step	Equation	Reason
0	$\mathbf{y} = \mathbf{f}(\mathbf{x}) = \dfrac{\mathbf{x}-1}{4\mathbf{x}+1}$	
1	$\mathbf{y} = \dfrac{\mathbf{x}-1}{4\mathbf{x}+1}$	
2	$\mathbf{y}(4\mathbf{x}+1) = \mathbf{x}-1$	
3	$4\mathbf{xy}+\mathbf{y} = \mathbf{x}-1$	
4	$4\mathbf{xy}-\mathbf{x} = -\mathbf{y}-1$	

5	$\mathbf{x}(4\mathbf{y}-1)=-\mathbf{y}-1$	
6	$\mathbf{x}=\dfrac{-\mathbf{y}-1}{4\mathbf{y}-1}=\dfrac{1+\mathbf{y}}{1-4\mathbf{y}}$	

Since $\mathbf{x}=\dfrac{1+\mathbf{y}}{1-4\mathbf{y}}$, we must exclude $\mathbf{y}=\dfrac{1}{4}$ since it makes the denominator zero: $\textbf{Range f}=\mathbb{R}_{\mathbf{y}}\setminus\left\{\dfrac{1}{4}\right\}=\left(-\infty,\dfrac{1}{4}\right)\cup\left(\dfrac{1}{4},+\infty\right)_{\mathbf{y}}$: