Algebra of FUNctions

Sum, Difference, Product, Quotient, and Composition of FUNctions

$$\left[\begin{array}{c}\text{MATH by Wilson} \\ \text{Your Personal Mathematics Trainer} \\ \text{MathByWilson.com}\end{array}\right]$$

We now create *new* functions from *old* (given) functions. Because finding their properties and graphs frequently requires techniques outside of those studied at this time, we just concentrate on finding their formulas and domains.

NEW FUNctions from OLD FUNctions:

Old: $\mathbf{f(x)}$; $\mathbf{g(x)}$; domains must be given or must be found

a. **New** – **Sum**: $\mathbf{f+g}$

$\mathbf{Dom\ (f+g) = Dom\ f \cap Dom\ g}$

$\mathbf{(f+g)(x) = f(x) + g(x)}$

b. **New** - **Difference**: $\mathbf{f-g\ ;\ g-f}$

$\mathbf{Dom\ (f-g) = Dom\ f \cap Dom\ g}$

$\mathbf{(f-g)(x) = f(x) - g(x)}$

c. **New** - **Product**: $\mathbf{fg}$

$\mathbf{Dom\ (fg) = Dom\ f \cap Dom\ g}$

$\mathbf{(fg)(x) = f(x)\ g(x)}$

d. **New** - **Quotient**: $\mathbf{\frac{f}{g}\ ;\ \frac{g}{f}}$

$\mathbf{Dom\ (\frac{f}{g}) = Dom\ f \cap \{x \in Dom\ g \mid g(x) \neq 0\}}$

$\mathbf{(\frac{f}{g})(x) = \frac{f(x)}{g(x)}}$

e. **New** - **Composition**: $\mathbf{f \circ g\ ;\ g \circ f}$

$\mathbf{Dom\ (f \circ g) = \{x \in Dom\ g \mid g(x) \in Dom\ f\}}$

$\mathbf{(f \circ g)(x) = f(g(x))}$

Since we studied sums, differences, products, and quotients of numbers many years ago, and functions are *just* collections of numbers, we study these new functions first.

Example 01: Consider the functions $\mathbf{f(x)}$ and $\mathbf{g(x)}$ defined in the following tables:

x	f(x)
-2	4
-1	2
0	1
3	5
5	4

x	g(x)
-3	-1
-1	5
0	3
2	-5
5	0

Find the Sum, Difference, Product, and Quotient of $\mathbf{f(x)}$ and $\mathbf{g(x)}$.

Solution:

We have

$-2 \quad -1 \quad 0 \quad 3 \quad 5$ Dom f

and

$-3 \quad -1 \quad 0 \quad 2 \quad 5$ Dom g

so that $\mathbf{Dom\ f} \cap \mathbf{Dom\ g} = \{-1, 0, 5\}$. Using the definitions above, we obtain

x	$(\mathbf{f}+\mathbf{g})(\mathbf{x})$	$(\mathbf{f}-\mathbf{g})(\mathbf{x})$	$(\mathbf{f\,g})(\mathbf{x})$	$\left(\frac{\mathbf{f}}{\mathbf{g}}\right)(\mathbf{x})$
-1	7	-3	10	2/5
0	4	-2	3	1/3
5	4	4	0	Undefined

For example,

$$(\mathbf{f}+\mathbf{g})(-1)=\mathbf{f}(-1)+\mathbf{g}(-1)=2+5=7$$
$$(\mathbf{f}-\mathbf{g})(0)=\mathbf{f}(0)-\mathbf{g}(0)=1-3=-2$$
$$(\mathbf{f}\,\mathbf{g})(5)=\mathbf{f}(5)*\mathbf{g}(5)=4*0=0$$
$$\left(\frac{\mathbf{f}}{\mathbf{g}}\right)(-1)=\frac{\mathbf{f}(-1)}{\mathbf{g}(-1)}=\frac{2}{5}$$
$$\left(\frac{\mathbf{f}}{\mathbf{g}}\right)(5)=\frac{\mathbf{f}(5)}{\mathbf{g}(5)}=\frac{4}{0}=\text{Undefined}$$

Example 02: Consider the functions $\mathbf{f}(\mathbf{x})$ and $\mathbf{g}(\mathbf{x})$ defined below:

$$\mathbf{f}(\mathbf{x})=\sqrt{3\mathbf{x}-2}\ ;\ \mathbf{g}(\mathbf{x})=\sqrt{4-5\mathbf{x}}$$

Find the Sum, Difference, Product, and Quotient of $\mathbf{f}(\mathbf{x})$ and $\mathbf{g}(\mathbf{x})$.

Solution:

First, note that

$$\textbf{Dom f}=\left[\frac{2}{3},+\infty\right)_{\mathbf{x}}\ ;\ \textbf{Dom g}=\left(-\infty,\frac{4}{5}\right]_{\mathbf{x}}\ :$$

___________[/ **Dom f**

$\frac{2}{3}$

/]__________ **Dom g**

$\frac{4}{5}$

___________[/ / / / / / / / / / / /]_________ **Dom f ∩ Dom g**

$\frac{2}{3}$ $\frac{4}{5}$

since $3\mathbf{x}-2\ge 0 \Rightarrow 3\mathbf{x}\ge 2 \Rightarrow \mathbf{x}\ge \frac{2}{3}$ and since $4-5\mathbf{x}\ge 0 \Rightarrow 4\ge 5\mathbf{x} \Rightarrow \frac{4}{5}\ge \mathbf{x} \Rightarrow \mathbf{x}\le \frac{4}{5}$.
Using our definitions, we have

Sum:

$$\mathbf{Dom}\left(\mathbf{f}+\mathbf{g}\right)=\left[\frac{2}{3},\frac{4}{5}\right]_{\mathbf{x}}$$

$$\left(\mathbf{f}+\mathbf{g}\right)(\mathbf{x})\equiv \mathbf{f}(\mathbf{x})+\mathbf{g}(\mathbf{x})=\sqrt{3\mathbf{x}-2}+\sqrt{4-5\mathbf{x}}$$; does NOT simplify ...

Difference:

$$\mathbf{Dom}\left(\mathbf{f}-\mathbf{g}\right)=\left[\frac{2}{3},\frac{4}{5}\right]_{\mathbf{x}}$$

$$\left(\mathbf{f}-\mathbf{g}\right)(\mathbf{x})\equiv \mathbf{f}(\mathbf{x})-\mathbf{g}(\mathbf{x})=\sqrt{3\mathbf{x}-2}-\sqrt{4-5\mathbf{x}}$$; does NOT simplify...

Product:

$$\mathbf{Dom}\left(\mathbf{f}\,\mathbf{g}\right)=\left[\frac{2}{3},\frac{4}{5}\right]_{\mathbf{x}}$$

$$\left(\mathbf{f}\,\mathbf{g}\right)(\mathbf{x})\equiv \mathbf{f}(\mathbf{x})\mathbf{g}(\mathbf{x})=\sqrt{3\mathbf{x}-2}\;\sqrt{4-5\mathbf{x}}$$

$$=\sqrt{\left(3\mathbf{x}-2\right)\left(4-5\mathbf{x}\right)}$$

$$=\sqrt{-15\mathrm{x}^2+22\mathbf{x}-8}$$; simplifies

Quotient:

$$\mathbf{Dom}\left(\frac{\mathbf{f}}{\mathbf{g}}\right)=\left[\frac{2}{3},\frac{4}{5}\right)_{\mathbf{x}}$$

$$\left(\frac{\mathbf{f}}{\mathbf{g}}\right)(\mathbf{x})\equiv \frac{\mathbf{f}(\mathbf{x})}{\mathbf{g}(\mathbf{x})}=\frac{\sqrt{3\mathbf{x}-2}}{\sqrt{4-5\mathbf{x}}}=\sqrt{\frac{3\mathbf{x}-2}{4-5\mathbf{x}}}$$; simplifies

Note: $\frac{4}{5}\notin \mathbf{Dom}\left(\frac{\mathbf{f}}{\mathbf{g}}\right)$ since $\mathbf{g}\left(\frac{4}{5}\right)=0$

Example 03: Consider the functions $\mathbf{f}(\mathbf{x})$ and $\mathbf{g}(\mathbf{x})$ defined below:

$$\mathbf{f}(\mathbf{x}) = \frac{\mathbf{x}}{\mathbf{x}-1} \quad ; \quad \mathbf{g}(\mathbf{x}) = \frac{\mathbf{x}+1}{\mathbf{x}}$$

Find the Sum, Difference, Product, and Quotient of $\mathbf{f}(\mathbf{x})$ and $\mathbf{g}(\mathbf{x})$.

Solution:

First, note that

$$\textbf{Dom f} = \mathbb{R}_{\mathbf{x}} \setminus \{1\} \ ; \ \textbf{Dom g} = \mathbb{R}_{\mathbf{x}} \setminus \{0\}$$

/ ∘ / / / / / / / / / / / / / / / / **Dom f**

1

/ / / / / / / / / / / / / / / / ∘ / **Dom g**

0

/ / / / / / / / / / / / / / / / ∘ / / / / / / / ∘ / / / / / / / / / / / / / / / / **Dom f** $\bigcap$ **Dom g**

0 1

We have

Sum:

$$\textbf{Dom}\left(\mathbf{f}+\mathbf{g}\right) = \mathbb{R}_{\mathbf{x}} \setminus \{0,1\}$$

$$\left(\mathbf{f}+\mathbf{g}\right)(\mathbf{x}) \equiv \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x}) = \frac{\mathbf{x}}{\mathbf{x}-1} + \frac{\mathbf{x}+1}{\mathbf{x}}$$

$$= \frac{\mathbf{x}^2 + \left(\mathbf{x}^2 - 1\right)}{\mathbf{x}\left(\mathbf{x}-1\right)} = \frac{2\mathbf{x}^2 - 1}{\mathbf{x}\left(\mathbf{x}-1\right)} \text{ ; simplifies}$$

Difference:

$$\mathbf{Dom}\left(\mathbf{f}-\mathbf{g}\right)=\mathbb{R}_{\mathbf{x}}\setminus\{0,1\}$$

$$\left(\mathbf{f}-\mathbf{g}\right)(\mathbf{x})\equiv\mathbf{f}(\mathbf{x})-\mathbf{g}(\mathbf{x})=\frac{\mathbf{x}}{\mathbf{x}-1}-\frac{\mathbf{x}+1}{\mathbf{x}}$$

$$=\frac{\mathbf{x}^2-\left(\mathbf{x}^2-1\right)}{\mathbf{x}\left(\mathbf{x}-1\right)}=\frac{1}{\mathbf{x}\left(\mathbf{x}-1\right)}\text{ ; simplifies}$$

Product:

$$\mathbf{Dom}\left(\mathbf{f}\,\mathbf{g}\right)=\mathbb{R}_{\mathbf{x}}\setminus\{0,1\}$$

$$\left(\mathbf{f}\,\mathbf{g}\right)(\mathbf{x})\equiv\mathbf{f}(\mathbf{x})\mathbf{g}(\mathbf{x})=\frac{\mathbf{x}}{\mathbf{x}-1}\frac{\mathbf{x}+1}{\mathbf{x}}\underset{(\mathbf{x}\neq0)}{=}\frac{\mathbf{x}+1}{\mathbf{x}-1}\text{ ; simplifies}$$

(Note: Do NOT simplify and then
find the domain)

Quotient:

$$\mathbf{Dom}\left(\frac{\mathbf{f}}{\mathbf{g}}\right)=\mathbb{R}_{\mathbf{x}}\setminus\{-1,0,1\}\ ;\mathbf{g}(-1)=0\Rightarrow\text{ Trash!}$$

$$\left(\frac{\mathbf{f}}{\mathbf{g}}\right)(\mathbf{x})\equiv\frac{\mathbf{f}(\mathbf{x})}{\mathbf{g}(\mathbf{x})}=\frac{\frac{\mathbf{x}}{\mathbf{x}-1}}{\frac{\mathbf{x}+1}{\mathbf{x}}}$$

$$\underset{(\mathbf{x}\neq0)}{=}\frac{\mathbf{x}}{\mathbf{x}-1}\ \frac{\mathbf{x}}{\mathbf{x}+1}=\frac{\mathbf{x}^2}{\mathbf{x}^2-1}\text{ ; simplifies}$$

The composition of two (2) functions is *different* but extremely important! Please be careful when using its definition.

Example 04: Consider the functions $\mathbf{f}(\mathbf{x})$ and $\mathbf{g}(\mathbf{x})$ defined in the following tables:

x	g(x)
-3	3
-1	-1
0	4
1	-2
2	3
4	1

x	f(x)
-3	-1
-1	0
0	-1
1	3
3	-2
4	5

Find the Composition $\mathbf{f}(\mathbf{x})$ with $\mathbf{g}(\mathbf{x})$: $(\mathbf{f} \circ \mathbf{g})(\mathbf{x})$

Solution:

We have

x	$(\mathbf{f} \circ \mathbf{g})(\mathbf{x})$
-3	-2
-1	0
0	5
2	-2
4	3

For example,

$-3 \in \mathbf{Dom}(\mathbf{f} \circ \mathbf{g})$ since $\mathbf{g}(3) = 3 \in \mathbf{Dom}\,\mathbf{f} \Rightarrow (\mathbf{f} \circ \mathbf{g})(-3) = \mathbf{f}(3) = -2$

$-1 \in \mathbf{Dom}(\mathbf{f} \circ \mathbf{g})$ since $\mathbf{g}(-1) = -1 \in \mathbf{Dom}\,\mathbf{f} \Rightarrow (\mathbf{f} \circ \mathbf{g})(-1) = \mathbf{f}(-1) = 0$

$0 \in \mathbf{Dom}(\mathbf{f} \circ \mathbf{g})$ since $\mathbf{g}(0) = 4 \in \mathbf{Dom}\,\mathbf{f} \Rightarrow (\mathbf{f} \circ \mathbf{g})(0) = \mathbf{f}(4) = 5$

$1 \notin \mathbf{Dom}(\mathbf{f} \circ \mathbf{g})$ since $\mathbf{g}(1) = -2 \notin \mathbf{Dom}\,\mathbf{f}$

Example 05: Consider the functions $\mathbf{f}(\mathbf{x})$ and $\mathbf{g}(\mathbf{x})$ defined below:

$$\mathbf{f}(\mathbf{x}) = 2\mathbf{x} - \mathbf{x}^2 \; ; \; \mathbf{g}(\mathbf{x}) = \mathbf{x}^2 + 2\mathbf{x} + 2$$

Find the Composition of $\mathbf{g}(\mathbf{x})$ with $\mathbf{f}(\mathbf{x})$ **and** $\mathbf{f}(\mathbf{x})$ with $\mathbf{g}(\mathbf{x})$:

Solution:

First, note that

$$\textbf{Dom f} = \mathbb{R}_{\mathbf{x}} \; ; \; \textbf{Dom g} = \mathbb{R}_{\mathbf{x}}$$

Then we have

Composition: $\mathbf{g} \circ \mathbf{f}$

$$\textbf{Dom}\left(\mathbf{g} \circ \mathbf{f}\right) = \left\{ \mathbf{x} \in \overbrace{\textbf{Dom f}}^{\mathbb{R}_{\mathbf{x}}} \;\middle|\; \mathbf{f}(\mathbf{x}) \in \overbrace{\textbf{Dom g}}^{\mathbb{R}_{\mathbf{x}}} \right\} = \mathbb{R}_{\mathbf{x}}$$

$$\begin{aligned}(\mathbf{g} \circ \mathbf{f})(\mathbf{x}) \equiv \mathbf{g}(\mathbf{f}(\mathbf{x})) &= [\mathbf{f}(\mathbf{x})]^2 + 2[\mathbf{f}(\mathbf{x})] + 2 \\ &= \left[2\mathbf{x} - \mathbf{x}^2\right]^2 + 2\left[2\mathbf{x} - \mathbf{x}^2\right] + 2 \\ &= 4\mathbf{x}^2 - 4\mathbf{x}^3 + \mathbf{x}^4 + 4\mathbf{x} - 2\mathbf{x}^2 + 2 \\ &= \mathbf{x}^4 - 4\mathbf{x}^3 + 2\mathbf{x}^2 + 4\mathbf{x} + 2\end{aligned}$$

and

Composition: $\mathbf{f} \circ \mathbf{g}$

$$\textbf{Dom}\left(\mathbf{f} \circ \mathbf{g}\right) = \left\{ \mathbf{x} \in \overbrace{\textbf{Dom g}}^{\mathbb{R}_{\mathbf{x}}} \;\middle|\; \mathbf{g}(\mathbf{x}) \in \overbrace{\textbf{Dom f}}^{\mathbb{R}_{\mathbf{x}}} \right\} = \mathbb{R}_{\mathbf{x}}$$

$$\begin{aligned}(\mathbf{f} \circ \mathbf{g})(\mathbf{x}) \equiv \mathbf{f}(\mathbf{g}(\mathbf{x})) &= 2[\mathbf{g}(\mathbf{x})] - [\mathbf{g}(\mathbf{x})]^2 \\ &= 2\left[\mathbf{x}^2 + 2\mathbf{x} + 2\right] - \left[\mathbf{x}^2 + 2\mathbf{x} + 2\right]^2 \\ &= 2\mathbf{x}^2 + 4\mathbf{x} + 4 - \left[\mathbf{x}^4 + 4\mathbf{x}^3 + 8\mathbf{x}^2 + 8\mathbf{x} + 4\right] \\ &= -\mathbf{x}^4 - 4\mathbf{x}^3 - 6\mathbf{x}^2 - 4\mathbf{x}\end{aligned}$$

Example 06: Consider the functions $\mathbf{f}(\mathbf{x})$ and $\mathbf{g}(\mathbf{x})$ defined below:

$$\mathbf{f}(\mathbf{x}) = \sqrt{\mathbf{x}-3} \;\; ; \;\; \mathbf{g}(\mathbf{x}) = \mathbf{x}^2 - 3\mathbf{x} - 25$$

Find the Composition of $\mathbf{f}(\mathbf{x})$ with $\mathbf{g}(\mathbf{x})$: $(\mathbf{f} \circ \mathbf{g})(\mathbf{x})$

Solution:

First, note that

$$\textbf{Dom f} = [3,+\infty)_{\mathbf{x}} \;\; ; \;\; \textbf{Dom g} = \mathbb{R}_{\mathbf{x}}$$

so that

Composition: $\mathbf{f} \circ \mathbf{g}$

$$\textbf{Dom}\,(\mathbf{f} \circ \mathbf{g}) = \left\{ \mathbf{x} \in \overbrace{\textbf{Dom g}}^{\mathbb{R}} \;\middle|\; \mathbf{g}(\mathbf{x}) \in \overbrace{\textbf{Dom f}}^{[3,+\infty)} \right\}$$

$$= (-\infty,-4] \cup [7,+\infty)$$

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$$-4 \qquad\qquad\qquad 7$$

To see this, note $\mathbf{g}(\mathbf{x}) \in [3,+\infty) \Rightarrow \mathbf{g}(\mathbf{x}) \ge 3 \Rightarrow \mathbf{x}^2 - 3\mathbf{x} - 25 \ge 3 \Rightarrow$

$$\mathbf{x}^2 - 3\mathbf{x} - 28 \ge 0 \Rightarrow (\mathbf{x}-7)(\mathbf{x}+4) \ge 0 \Rightarrow \mathbf{x} \in (-\infty,-4] \cup [7,+\infty)$$

$$\therefore \textbf{Dom}(\mathbf{f} \circ \mathbf{g}) = (-\infty,-4] \cup [7,+\infty)$$

Also

$$(\mathbf{f} \circ \mathbf{g})(\mathbf{x}) \equiv \mathbf{f}(\mathbf{g}(\mathbf{x})) = \sqrt{[\mathbf{g}(\mathbf{x})] - 3}$$

$$= \sqrt{\left[\mathbf{x}^2 - 3\mathbf{x} - 25\right] - 3}$$

$$= \sqrt{\mathbf{x}^2 - 3\mathbf{x} - 28}$$

