Linear Functions - Equations of Lines

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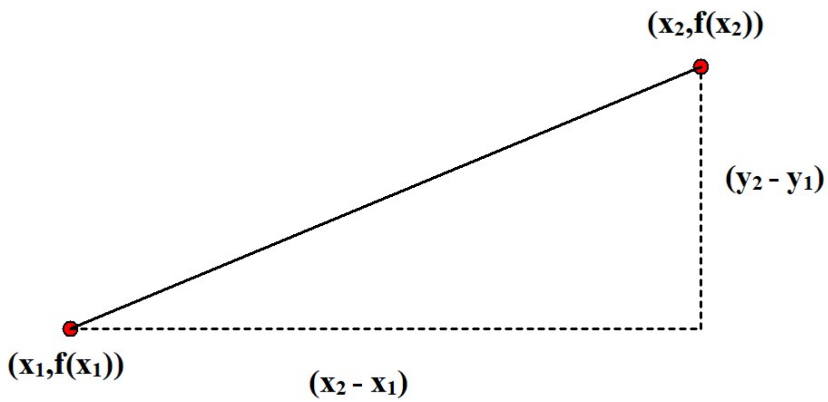
Linear Function Form (Slope & y-Intercept): $\mathbf{y}=\mathbf{f}(\mathbf{x})=\mathbf{mx}+\mathbf{b}$

If $\mathbf{x}=0 \Rightarrow \mathbf{y}=\mathbf{f}(0)=\mathbf{m}*0+\mathbf{b}=\mathbf{b} \Rightarrow (0,\mathbf{b})$ is the y-intercept point.

The letter **m** represents the “slope” of the “line” formed by $\mathbf{y}=\mathbf{f}(\mathbf{x})=\mathbf{mx}+\mathbf{b}$. For if we pick two different points $(\mathbf{x}_1,\mathbf{y}_1)$ & $(\mathbf{x}_2,\mathbf{y}_2)$ on the graph of $\mathbf{f}$ and find $\frac{\text{Change in the y-values}}{\text{Change in the x-values}}$, we obtain

$$\begin{aligned}\frac{\text{Change in the y-values}}{\text{Change in the x-values}} &= \frac{\mathbf{f}(\mathbf{x}_2)-\mathbf{f}(\mathbf{x}_1)}{\mathbf{x}_2-\mathbf{x}_1} \\ &= \frac{(\mathbf{mx}_2+\mathbf{b})-(\mathbf{mx}_1+\mathbf{b})}{\mathbf{x}_2-\mathbf{x}_1} \\ &= \frac{\mathbf{m}(\mathbf{x}_2-\mathbf{x}_1)}{(\mathbf{x}_2-\mathbf{x}_1)} \\ &= \mathbf{m}\end{aligned}$$

Slope



So, for any two points, the $\dfrac{\text{Change in the y-values}}{\text{Change in the x-values}}$ is ALWAYS the same constant **m**. Therefore, its graph is a straight line with slope **m**. If $\mathbf{m} = 0$, we obtain a horizontal line: $\mathbf{y} = \mathbf{b}$

Consider $\mathbf{y} = \mathbf{f}(\mathbf{x}) = \mathbf{mx} + \mathbf{b} = -\frac{3}{2}\mathbf{x} + 4$. We'll now find four (4) important properties that **f** possesses.

Properties:

1. Domain: $\textbf{Dom f} = \mathbb{R}_{\mathbf{x}}$; there are no real numbers to reject.

2. Intercepts:

   a. $\mathbf{y}$: Set $\mathbf{x}=0$

      $\mathbf{f}(0)=4 \Rightarrow (0,4)=(0,\mathbf{b})$

   b. $\mathbf{x}$: Set $\mathbf{y}=\mathbf{f}(\mathbf{x})=0$

      $0 \overset{\text{SET}}{=} \mathbf{y}=\mathbf{f}(\mathbf{x})=-\frac{3}{2}\mathbf{x}+4$

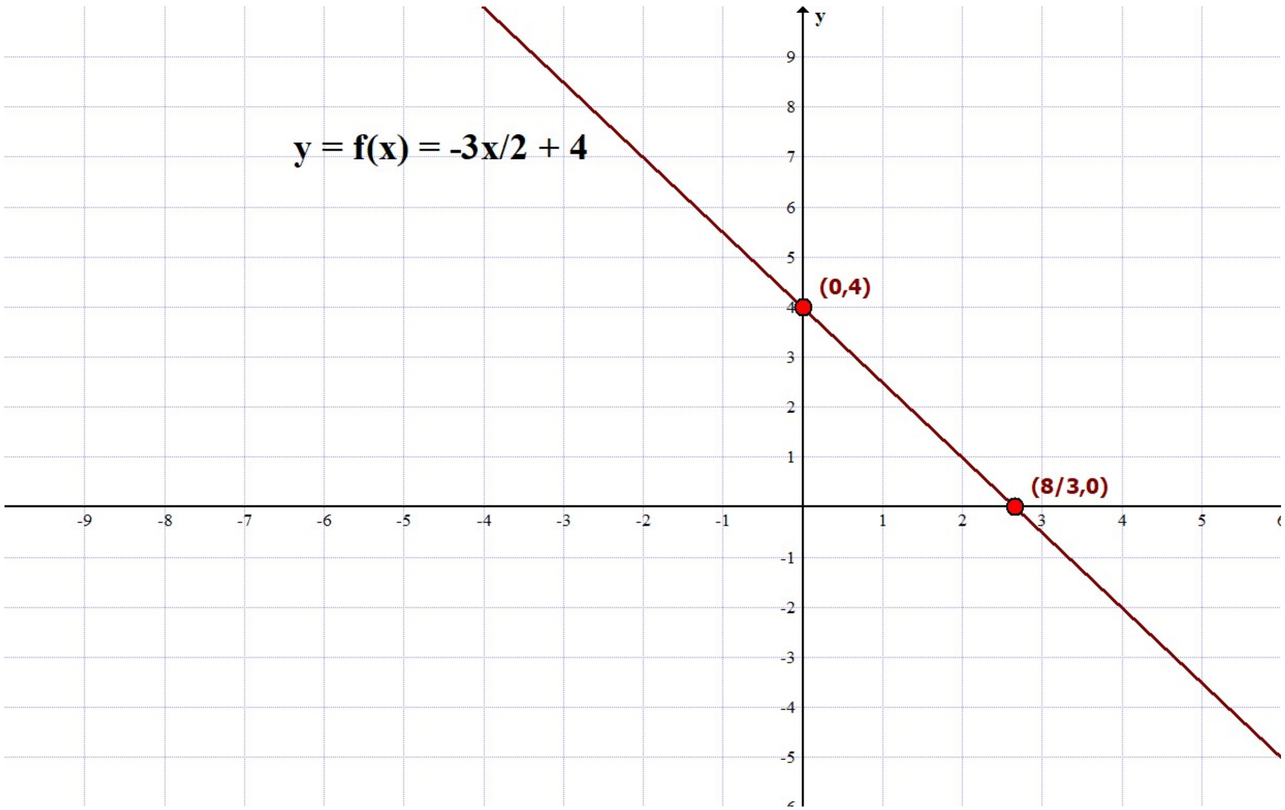
      $\frac{3}{2}\mathbf{x}=4$

      $\mathbf{x}=\frac{8}{3} \Rightarrow \left(\frac{8}{3},0\right)$; **x**-intercept point

3. Slope: $\mathbf{m}=-\frac{3}{2}$

4. Range: $\textbf{Range f}=\mathbb{R}_{\mathbf{y}}$, note the **range** is the projection of the graph onto the y-axis.

Drawing a straight line through the two (2) intercept points, we obtain the graph of **f**:



Although a straight line may be represented by $\mathbf{y = mx + b}$, there are several other ways to represent a line:

Given two (2) points $\mathbf{P}(\mathbf{x_1, y_1})$ and $\mathbf{Q}(\mathbf{x_2, y_2})$ (so that the slope is $\mathbf{m} = \frac{\mathbf{y_2 - y_1}}{\mathbf{x_2 - x_1}}$) **or** one point $\mathbf{P}(\mathbf{x_1, y_1})$ and a slope $\mathbf{m}$, there are four (4) forms the equation of a straight line using these data can take:

1. **Standard Form**: $\mathbf{Ax}+\mathbf{By}=\mathbf{C}$
    a. Two Points: $\mathbf{A}=\mathbf{y}_2-\mathbf{y}_1$; $\mathbf{B}=\mathbf{x}_1-\mathbf{x}_2$; $\mathbf{C}=\mathbf{x}_2\mathbf{y}_1-\mathbf{x}_1\mathbf{y}_2$
    b. Point & Slope: $\mathbf{A}=\mathbf{m}$; $\mathbf{B}=-1$; $\mathbf{C}=\mathbf{mx}_1-\mathbf{y}_1$
2. **Slope & y-Intercept Form**: $\mathbf{y}=\mathbf{mx}+\mathbf{b}$; $\mathbf{m}=\dfrac{\mathbf{y}_2-\mathbf{y}_1}{\mathbf{x}_2-\mathbf{x}_1}$; $\mathbf{b}=\mathbf{y}_1-\dfrac{\mathbf{y}_2-\mathbf{y}_1}{\mathbf{x}_2-\mathbf{x}_1}\mathbf{x}_1$ (Linear Function Form)
3. **Point & Slope Form**: $\mathbf{y}-\mathbf{y}_1=\mathbf{m}(\mathbf{x}-\mathbf{x}_1)$; $\mathbf{m}=\dfrac{\mathbf{y}_2-\mathbf{y}_1}{\mathbf{x}_2-\mathbf{x}_1}$ if two points are given
4. **Two Point Form**: $\mathbf{y}-\mathbf{y}_1=\dfrac{\mathbf{y}_2-\mathbf{y}_1}{\mathbf{x}_2-\mathbf{x}_1}(\mathbf{x}-\mathbf{x}_1)$

Example 01: Consider the straight line $4\mathbf{x}+7\mathbf{y}=24$. Put this line in "linear function form" and find the properties of the function $\mathbf{f}$.
Solution:

We first solve $4\mathbf{x}+7\mathbf{y}=24$ for $\mathbf{y}$:

Step	Equation	Reason
0	$4\mathbf{x}+7\mathbf{y}=24$	
1	$7\mathbf{y}=24-4\mathbf{x}$	
2	$\mathbf{y}=\dfrac{24-4\mathbf{x}}{7}=-\dfrac{4}{7}\mathbf{x}+\dfrac{24}{7}$	
3	$\mathbf{y}=\mathbf{f}(\mathbf{x})=-\dfrac{4}{7}\mathbf{x}+\dfrac{24}{7}$ $=\mathbf{mx}+\mathbf{b}$	

Properties:

1. Domain: $\mathbf{Dom\ f} = \mathbb{R}_{\mathbf{x}}$

2. Intercepts:

   a. $\mathbf{y}$: Set $\mathbf{x} = 0$

   $$\mathbf{f}(0) = \frac{24}{7} \Rightarrow \left(0, \frac{24}{7}\right) = (0, \mathbf{b})$$

   b. $\mathbf{x}$: Set $\mathbf{y} = \mathbf{f}(\mathbf{x}) = 0$

   $$0 \overset{\text{SET}}{=} \mathbf{y} = \mathbf{f}(\mathbf{x}) = -\frac{4}{7}\mathbf{x} + \frac{24}{7}$$

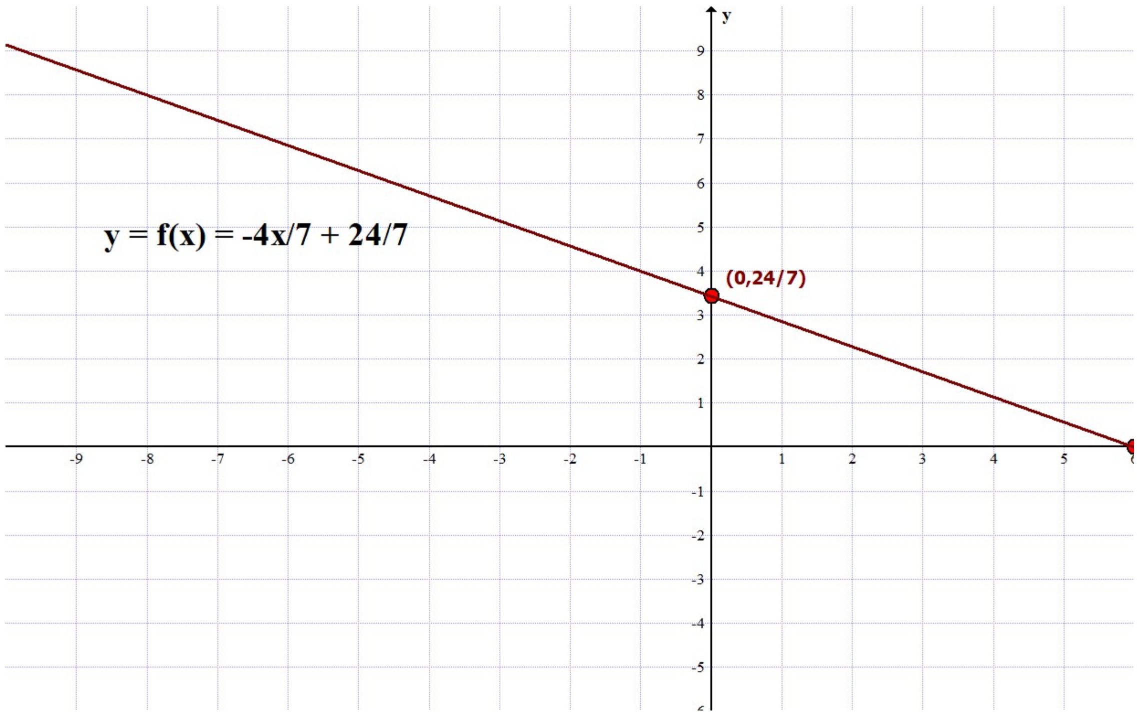
   $$\frac{4}{7}\mathbf{x} = \frac{24}{7}$$

   $$\mathbf{x} = \frac{24}{4} = 6 \Rightarrow (6, 0)$$

3. Slope: $\mathbf{m} = -\frac{4}{7}$

4. Range: $\mathbf{Range\ f} = \mathbb{R}_{\mathbf{y}}$

Below, we have the graph of $\mathbf{f}$:



Example 02: Consider the straight line defined by $\mathbf{m} = \frac{5}{3}$ and $\mathbf{P}(-4,7)$. Put this line in "linear function form" and find the properties of the function $\mathbf{f}$.

Solution:

Using the Point & Slope Form, we obtain:

Step	Calculation	Reason
0	$\mathbf{y}-\mathbf{y}_1=\mathbf{m}(\mathbf{x}-\mathbf{x}_1)$	
1	$\mathbf{y}-[7]=\left[\frac{5}{3}\right](\mathbf{x}-[-4])$	
2	$\mathbf{y}=\frac{5}{3}\mathbf{x}+\frac{20}{3}+7=\frac{5}{3}\mathbf{x}+\frac{41}{3}$	
3	$\mathbf{y}=\mathbf{f}(\mathbf{x})=\frac{5}{3}\mathbf{x}+\frac{41}{3}$ $=\mathbf{mx}+\mathbf{b}$	

Properties:

1. Domain: $\mathbf{Dom\ f}=\mathbb{R}_{\mathbf{x}}$

2. Intercepts:

   a. $\mathbf{y}$: Set $\mathbf{x}=0$

   $$\mathbf{f}(0)=\frac{41}{3}\Rightarrow\left(0,\frac{41}{3}\right)=(0,\mathbf{b})$$

   b. $\mathbf{x}$: Set $\mathbf{y}=\mathbf{f}(\mathbf{x})=0$

   $$\mathbf{y}=\mathbf{f}(\mathbf{x})=\frac{5}{3}\mathbf{x}+\frac{41}{3}=0$$

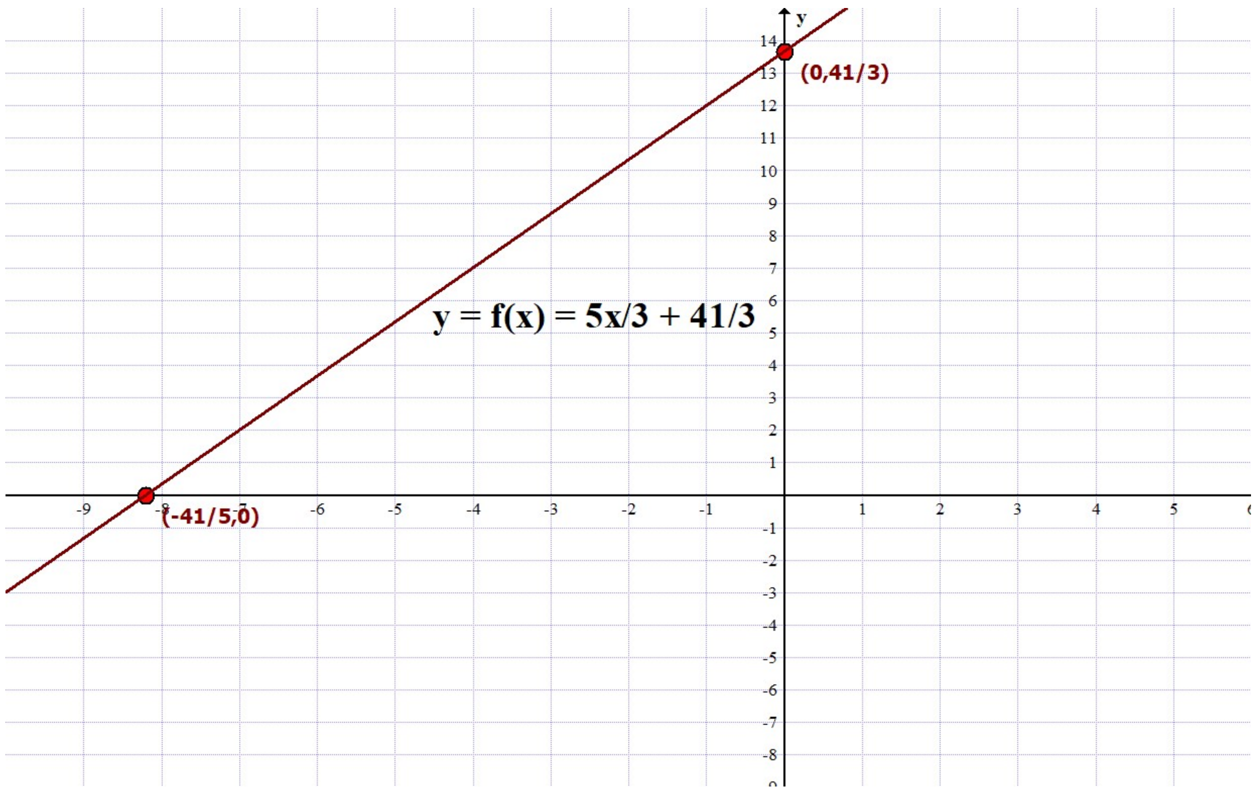
   $$\frac{5}{3}\mathbf{x}=-\frac{41}{3}$$

   $$\mathbf{x}=-\frac{41}{5}\Rightarrow\left(-\frac{41}{5},0\right)$$

3. Slope: $\mathbf{m} = \frac{5}{3}$; given

4. Range: $\textbf{Range f} = \mathbb{R}_{\mathbf{y}}$

The graph is below:



Example 03: Consider the straight line defined by $\mathbf{P}(-3,2)$ and $\mathbf{Q}(5,9)$. Put this line in "linear function form" and find the properties of the function $\mathbf{f}$.

Solution:

The slope is $\mathbf{m}=\frac{\mathbf{y}_2-\mathbf{y}_1}{\mathbf{x}_2-\mathbf{x}_1}=\frac{[9]-[2]}{[5]-[-3]}=\frac{7}{8}$. Now, using the Point & Slope Form, we have:

Step	Calculation	Reason
0	$\mathbf{y}-\mathbf{y}_1=\mathbf{m}(\mathbf{x}-\mathbf{x}_1)$	
1	$\mathbf{y}-[2]=\left[\frac{7}{8}\right](\mathbf{x}-[-3])$	
2	$\mathbf{y}=\frac{7}{8}\mathbf{x}+\frac{21}{8}+2=\frac{7}{8}\mathbf{x}+\frac{37}{8}$	
3	$\mathbf{y}=\mathbf{f}(\mathbf{x})=\frac{7}{8}\mathbf{x}+\frac{37}{8}$	

Properties:

1. Domain: $\textbf{Dom f}=\mathbb{R}_{\mathbf{x}}$

2. Intercepts:

   a. $\mathbf{y}$: Set $\mathbf{x}=0$

   $\mathbf{f}(0)=\frac{37}{8}\Rightarrow\left(0,\frac{37}{8}\right)=(0,\mathbf{b})$

b. $\mathbf{x}$: Set $\mathbf{y} = \mathbf{f}(\mathbf{x}) = 0$

$$\mathbf{y} = \mathbf{f}(\mathbf{x}) = \frac{7}{8}\mathbf{x} + \frac{37}{8} \overset{\text{SET}}{=} 0$$

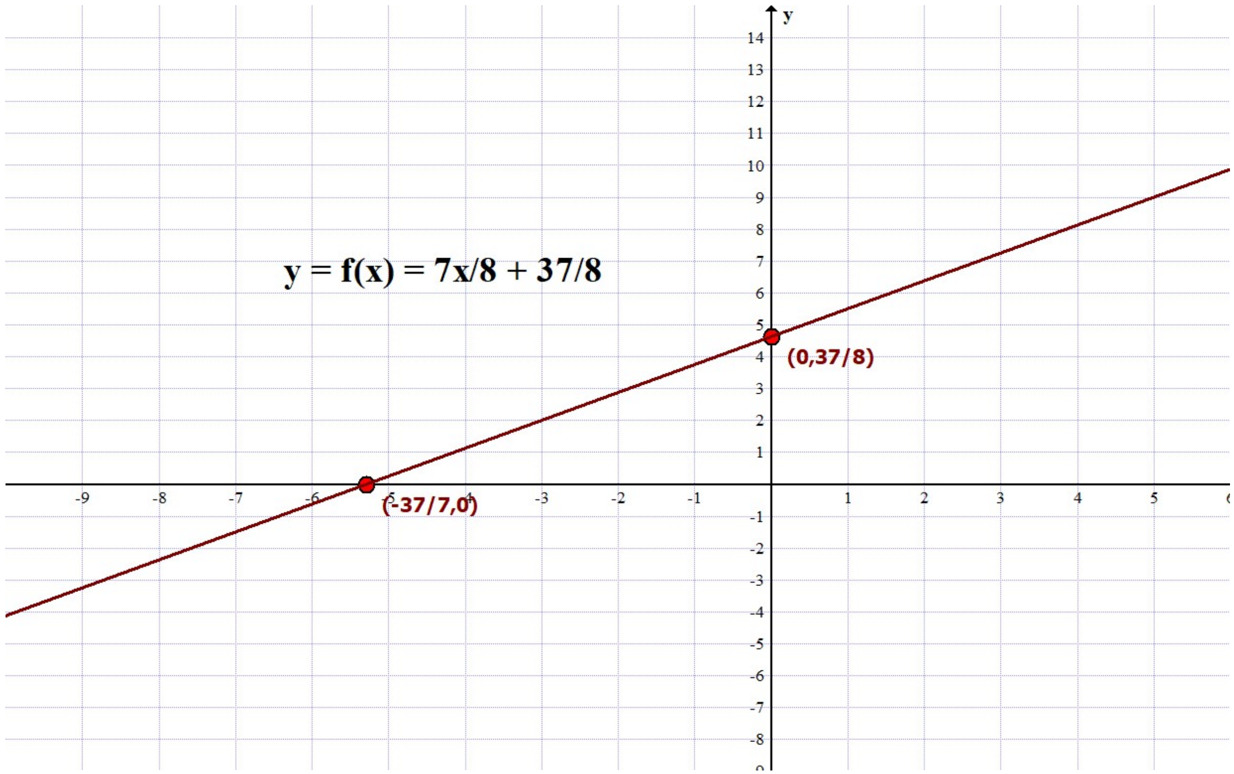
$$\frac{7}{8}\mathbf{x} = -\frac{37}{8}$$

$$\mathbf{x} = -\frac{37}{7} \Rightarrow \left(-\frac{37}{7}, 0\right)$$

3. Slope: $\mathbf{m} = \frac{7}{8}$

4. Range: $\textbf{Range f} = \mathbb{R}_{\mathbf{y}}$

The graph is below:


y = f(x) = 7x/8 + 37/8
(0,37/8)
(-37/7,0)
y

