Real Number System

Natural, Whole Numbers, Integers

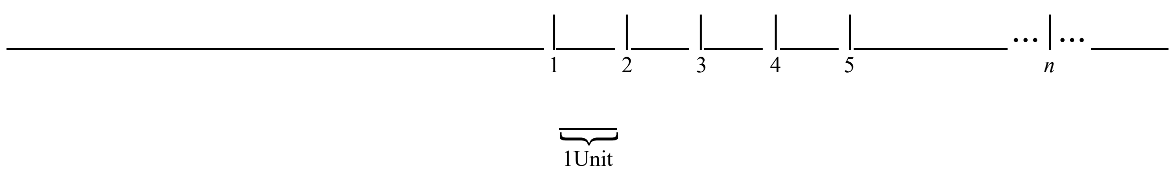
(Types, Number Line, Equality, Inequality, Operations, Properties, ...)

N = Natural Numbers

These are also called the **counting numbers** and are represented by the set

N = {1,2,3,4,5,6,7, ..., n, ...}

where "n" represents an element (member) of the set. Geometrically, we use a "**number line**" to illustrate these numbers:

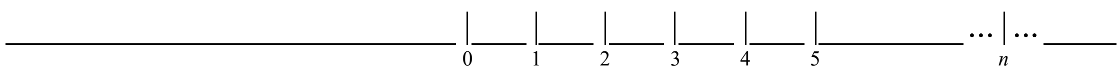


We use the braces symbol {...} to enclose the members (elements) of our set **N**. Obviously, these numbers do NOT fill up the number line so we will keep introducing new numbers until every point on this line is associated with a number. This set of *all* numbers that fill up the number line is called the set of **real numbers**.

W = Whole Numbers

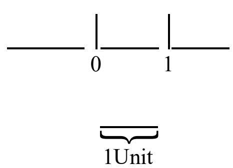
We just add “0” to the set of natural numbers.

Geometrically, we have



All the numbers to the right of “0” are called **positive numbers** and technically have a “+” sign in front of them that we do NOT have to write; it’s optional. For example, we can write +4 for 4, that is + 4 = 4.

By definition, all of the whole numbers are one unit apart:



In application problems, this length can be inches, feet, yards, miles, … The numbers 0,1,2,3,4,5,6,7,8,9 are *one digit* numbers and are called **units**:

0

1 = 0 + 1

2 = 1 + 1

3 = 2 + 1

.

.

.

9 = 8 + 1

This notation is our first *taste* of the concept of **addition (“+”)**. Next on the number line we have the **tens**:

10 = 9 + 1

11 = 10 + 1

12 = 11 + 1

.

.

.

98 = 97 + 1

99 = 98 + 1

To the right of the tens on the number line are the **hundreds**:

100 = 99 + 1

101 = 100 + 1

102 = 101 + 1

.

.

.

998 = 997 + 1

999 = 998 + 1

The **thousands** are next:

1000 = 999 + 1

1001 = 1000 + 1

1002 = 1001 + 1

.

.

.

9998 = 9997 + 1

9999 = 9998 + 1

Then the **ten thousands:**

10000 = 9999 + 1

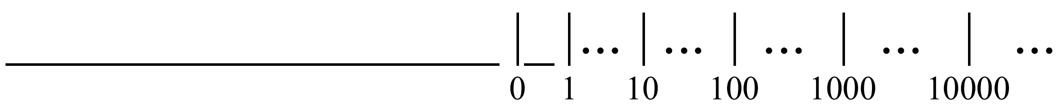
10001 = 10000 + 1

10002 = 10001 + 1

.

.

.



We can classify the whole numbers as follows:

…	Thousands	Hundreds	Tens	Units
				7
			3	2
		6	4	5
	8	2	1	4

We have, for example, 8214 = 8000 + 200 + 10 + 4. The numbers 8, 2, 1, 4 are called **place values**:

1. 8 is the place values for the thousands
2. 2 is the place value for the hundreds
3. 1 is the place value for the tens
4. 4 is the units

Note

We are using the letter "n" to represent numbers that can take on different natural, whole, or integer values. They are called **variables** since they are allowed to change. In general, we will use other letters such as "a", "b", "c" when we list the properties that *all* our "real" numbers possess.

Using letters to represent numbers is the "key" to algebra which we now define:

Algebra – arithmetic with letters or applied arithmetic.

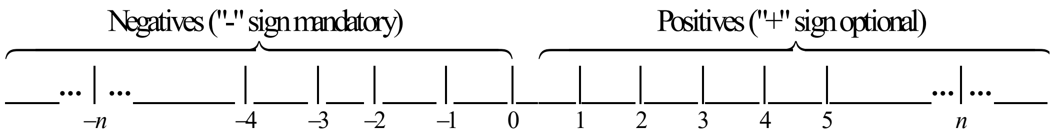
Hence, to be successful in algebra, we need to have mastered arithmetic.

I = Integers

Now we add the "negatives" (opposites) of the set of natural numbers:

$$\{\ldots,-n,\ldots,-7,-6-5,-4,-3,-2,-1\}$$

Geometrically, we have



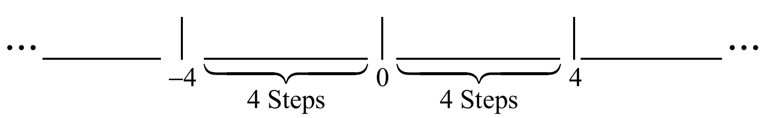
The number "0" is neither positive or negative.

Size and Direction:

Numbers have two (2) characteristics we now discuss:

1. Size or Magnitude
2. Direction

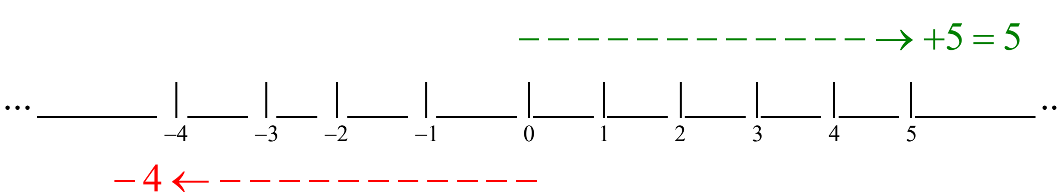
The **size or magnitude** of a number "a" is measured by its **absolute value:** $|a|$. For integers, it's the number of steps we must take from "0" to get to the integer. Obviously, $|0| = 0$. Also, for example, $|-4| = \overset{\text{Steps}}{4} = |4|$:



For now, we just omit the "-" if there is one: $|-5| = 5$.

The **direction** is determined by the "+" or "-" sign:

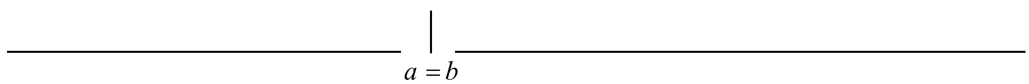
1. The "+" sign means to the "**right**" of "0".
2. The "-" sign means to the "**left**" of "0".



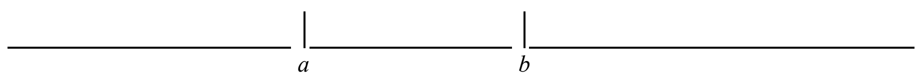
Notice that we can think of the integers as “arrows” starting at “0”.

Equality and Inequality (that is**, Non-equality)**

When two (2) numbers “ a & b ” are the same, we write $a = b$; obviously, $6 = 6$.



If they are NOT equal, we write $a \neq b$; for example, $6 \neq 8$. If they are NOT the same, one of them is to the left of the other one on the number line. If the number “a” is the one on the left, we write $a < b$ and say that “a **is less than** b”. We may also write $b > a$ and say “b **is greater than** a”.



a < b

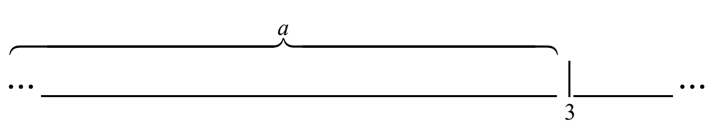
OR

b > a

We write $a \leq b$ whenever “a **is less than OR equal to** b ” and $b \geq a$ whenever “ b **is greater than OR equal to** a ”. Using our new symbols $<$; $>$ $\leq$; $\geq$, we have

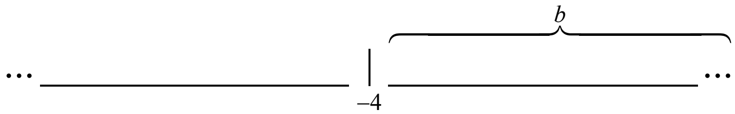
$$-3 < 2 \ ; \ 2 \leq 5 \ ; -4 \leq 4 \ ; \ -1 > -2 \ ; \ 3 \geq 3$$

We can graph these numbers on a number line to verify that these statements are true. If "a" *is less than* 3, we write $a < 3$ and draw a picture (graph) on the number line:



Hence, “a” can be *any* number less than (to the left of) 3.

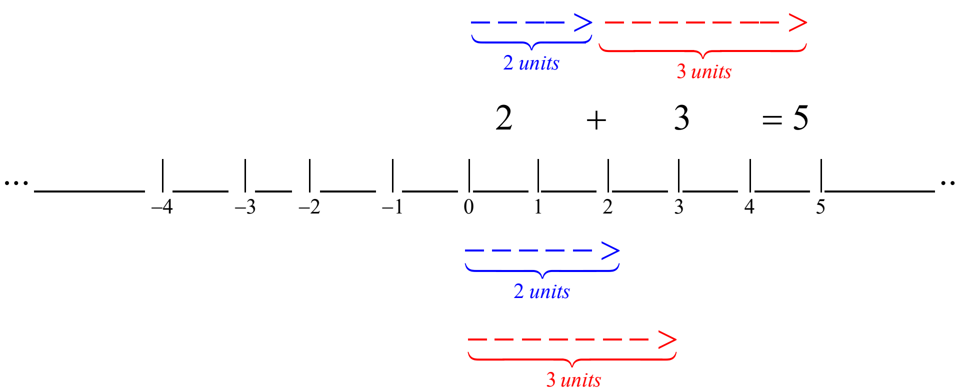
If "b" *is greater than* -4, we write $b > -4$ and draw



Hence, "b" can be any number greater than (to the right of) – 4.

Operations on numbers: New numbers from given numbers

Addition: Given two (2) numbers "a & b", called **addends**, we define its **sum**, denoted $a \underbrace{+}_{\substack{\text{Addition}\\ \text{Sign}}} b$, using our geometrical interpretation of the numbers. For example, 2 + 3 = 5:



Notice that 3 + 2 = 5 and this illustrates a truth about the **sum** of two numbers, namely, $a + b = b + a$.

Note

a + b:

"a" and "b" are called **addends.**

"$a + b$" is called the **Sum.**

$a + b$ is a unique number. This is called **closure.**

The numbers "a" and "b" are called **terms** because they are associated with **addition**.

Communicative Property with respect to Addition: If "a" and "b" are numbers, then

a + b = b + a

Working temporarily with only positive integers and zero, we may use our new definition of addition to build the following table:

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

For example, 3 + 6 = 9, 5 + 7 = 12, … Note that the sum of two (2) numbers is unique (only one number). PLEASE understand that the "+" sign is used as the "addition sign" **and** was used previously to tell us that a number is positive: +4 = 4.

If we have three numbers "$a, b, \& c$", we can form $a + (b + c)$ or $(a + b) + c$. Fortunately, they represent the same number, as the following numbers illustrate: $2 + (4 + 3) = 2 + 7 = 9$ and $(2 + 4) + 3 = 6 + 3 = 9$. Thus, we are allowed to write $a + (b + c) = a + b + c = (a + b) + c$

In addition, we have

3 + 0 = 3 = 0 + 3.

In general, we have a + 0 = a = 0 + a and "0" is called the **additive identity.**

Also, note that (- 4) + (+ 4) = 0 = (+ 4) + (- 4).

In general, we have a + (- a) = 0 = (- a) + a. The number “- a” is called the **additive inverse** (opposite) of “a” and the number “- (- a)” = a is called the **additive inverse** (opposite) of “- a”.

Note

Associative Property with respect to Addition:

For three numbers “ $a, b, \& c$ ”, we have

a + (b + c) = (a +b) + c

Identity Element with respect to Addition: “0”

For any number “ a ”, we have

a + 0 = a = 0 + a

Additive Inverse:

For any number “ a ”, we have

a + (- a) = 0 = (- a) + a

and

(-a) – [- (- a)] = 0 = [- (- a)] – (-a)

The **Additive Inverse** of **a** is (**-a**) and the **Additive Inverse** of (**-a**) is **–(-a)** = **a**

Subtraction: Addition in reverse

The operation of addition has an inverse (opposite) called **subtraction**:

$$a \underbrace{-}_{\text{Subtraction Sign}} b.$$

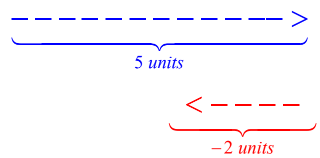
Note that “a” is called the **minuend**, “b” is called the **subtrahend**, and $a - b$ is called the **difference** of “a” and “b”.

Subtraction is defined by

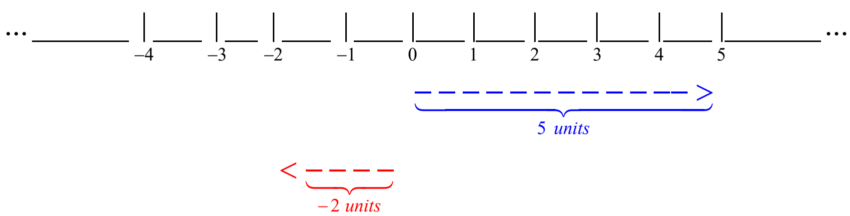
$$a-b \overset{Defined}{\equiv} a+(-b)$$

so that, for example,

5 – 2 = 5 + (- 2) = 3



$$5-2=5+(-2)=3$$



Also, for example,

4 – 7 = 4 + (- 7) = -3

- 2 – (- 4) = - 2 + 4 = 2

1 – (- 3) = 1 + 3 = 4

Note

a - b:
“a” is called the **minuend**
“b” is called the **subtrahend**

"$a-b$" is called the **difference:** $a-b \overset{Defined}{\equiv} a+(-b)$

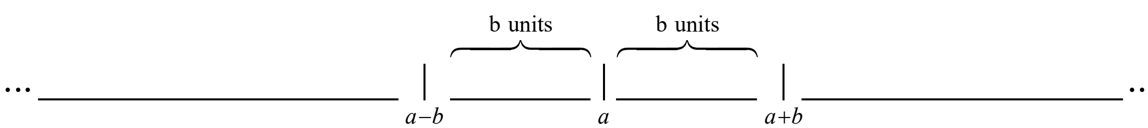
$a-b$ is a unique number. This is called **closure.**

The numbers “a” and “b” are called **terms** because they are associated with **subtraction**.

Note: The symbol “-“ is used *both* for the direction of integers AND the subtraction of integers.

Note: $8-3=5\neq -5=3-8$; in general, $a-b\neq b-a$. Recall, $a+b=b+a$ so that addition and subtraction do NOT have the same properties.

Note: $a+b$ and $a-b$ are the same number of units (“steps”) from “a” but in opposite (inverse) directions. Assuming b > 0, we have



Multiplication: Shorthand notation for addition

Let’s say we want to add the number 3 to itself 7 times:

3 + 3 + 3 + 3 + 3 + 3 + 3 = 21

The symbols $7*3=7\bullet 3=7\times 3=21$ are *all* shorthand for adding 3 to itself 7 times and are said to be 7 **multiplied** by 3. By the way, we will NOT use the last one, that is “x”, for multiplication. We can see that we get 21 geometrically by just adding the number of squares:

	7						
	1	2	3	4	5	6	7
3	8	9	10	11	12	13	14
	15	16	17	18	19	20	21

In 7*3 = 21, the numbers “7” and “3” are called **factors** and “21” is called the **product** of 7 and 3.

Notice that we also get 21 by switching the “3” and the “7”:

3

7

1	**2**	**3**
4	**5**	**6**
7	**8**	**9**
10	**11**	**12**
13	**14**	**15**
16	**17**	**18**
19	**20**	**21**

Thus, 7*3 = 21 = 3*7; in general a*b = b*a

Note

a*b:

"a" and "b" are **factors**

"a*b" is the **product**

a*b is a unique number. This is called **closure.**

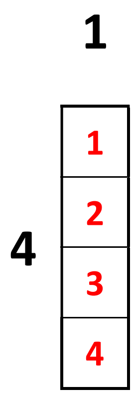
Commutative Property with respect to Multiplication:

For two numbers "$a \& b$", we have

a * b = b * a

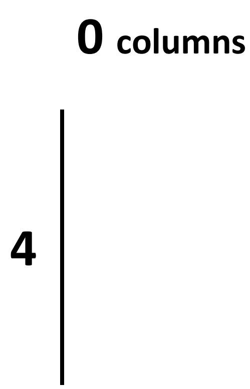
also written as **a b = b a**

There are two (2) special cases to consider. First, notice that



so that 4*1 = 4 ; in general, a*1 = a = 1*a . The number "1" is called the **multiplicative identity**.

Also, 4*0 = 0:



so that 4*0 = 0 ; in general, a*0 = 0 = 0*a

Note

> **Identity Element with respect to Multiplication: "1"**
>
> For any number "a", we have
>
> **a *1 = a = 1* a**

Zero Property with respect to Multiplication: "0"

For any number " a ", we have

a *0 = 0 = 0* a

Unless one or both of the numbers "a" and "b" are "0", they will be either be positive (> 0) or negative (< 0). Here are the "**sign rules**" for multiplying "a" and "b":

a	B	a*b
(+3) ; Positive	(+4) ; Positive	= + 12 = 12 ; **Positive**
(+5) ; Positive	(- 2) ; Negative	= - 10 ; **Negative**
(- 4) ; Negative	(+6) ; Positive	= - 24 ; **Negative**
(- 2) ; Negative	(- 4) ; Negative	= + 8 = 8 ; **Positive**

Just as a + (b + c) = (a + b) + c, we have a * (b * c) = (a * b) * c

as 2*(3*5) = 2*15 = 30 and (2*3)*5 = 6*5 = 30 illustrate. We can write a*b*c since a * (b * c) = (a * b) * c.

There is one more *important property* we need to point out:

2*(3 + 5) = 2 + 8 = 16

is the same as

2*3 + 2*5 = 6 + 10 = 16.

The general property is a*(b + c) = a*b + a*c. Also true is a*(b - c) = a*b - a*c.

For example,

5*(2 + 3) = 5*5 = 25

5*2 + 5*3 = 10 + 15 = 25

and

5*(2 - 3) = 5*(-1) = - 5

5*2 - 5*3 = 10 - 15 = - 5

Note

Distributive Property with respect to Addition/Subtraction and Multiplication:

For three numbers " $a, b \& c$ ", we have

a * (b + c) = a*b + a*c

a * (b - c) = a*b - a*c

The **Distributive Property** is extra important because it contains two (2) operations.

Exponentiation: Shorthand for multiplication

Whereas multiplication is a shorthand notation for addition, **exponentiation** is a shorthand notation for multiplication. We have considered

$$a \underbrace{+}_{\substack{\text{Addition}\\ \text{Sign}}} b \,;\, a \underbrace{-}_{\substack{\text{Subtraction}\\ \text{Sign}}} b \,;\, a \underbrace{*}_{\substack{\textit{Multiplication}\\ \text{Sign}}} b\,.$$

Now we consider numbers of the form

$$b^n = \text{Base}^{\text{Exponent}}$$

where, for now, "b" is an integer and "n" is a positive integer. This means that "b" is used as a factor "n" times:

$$b^n = \overbrace{b * b * b * \ldots * b}^{\text{n times}}$$

For example,

$$2^3 = 2*2*2 = 8$$
$$3^2 = 3*3 = 9$$
$$1^5 = 1*1*1*1*1 = 1$$
$$(-2)^4 = (-2)*(-2)*(-2)*(-2) = 16$$
$$(-2)^3 = (-2)*(-2)*(-2) = -8$$

These new numbers are often called **power numbers** (numbers to a power (exponent)).

We define $b^0 = 1 \; ; \; b \neq 0$:

$$(+3)^0 = 1$$
$$(-5)^0 = 1$$

There are a lot of properties that power numbers have; here are a couple of them with illustrations:

1. $(a^n)^m = a^{n*m}$

   Example: $(2^3)^2 = 8^2 = 64$ & $2^{3*2} = 2^6 = 2*2*2*2*2*2 = 64$

2. $(a*b)^n = a^n * b^n$

   Example: $(2*5)^3 = 10^3 = 1000$ & $2^3 * 5^3 = 8*125 = 1000$

Note

$$b^n \overset{Defined}{\equiv} \overbrace{b*b*b*...*b}^{\text{n times}}$$

Definition: $b^0 = 1 \; ; \; b \neq 0$

Basic Exponential Properties:

For two integers "$a \& b$", we have

1. $(a^n)^m = a^{n*m}$
2. $(a*b)^n = a^n * b^n$

Division: multiplication in reverse

Consider the number 12. Geometrically, we can **divide** 12 into 3 pieces of 4 *OR* 4 pieces of 3:

1	**2**	**3**	**4**

1	**2**	**3**	**4**

1	**2**	**3**	**4**

1	**2**	**3**

1	**2**	**3**

1	**2**	**3**

1	**2**	**3**

Since 12 can be divided into 4 equal units of 3, we write $\frac{12}{4} = 3$ and say that **12 divided by 4 is 3**. We also can write $\frac{12}{3} = 4$ and say that **12 divided by 3 is 4.**

For now, we are just considering *positive* integers and whenever we have c = a*b, we can write

1. $\frac{c}{a} = b$; **"c" divided by "a" is "b"**
2. $\frac{c}{b} = a$ **"c" divided b "b" is "a"**

Given that $\frac{c}{a} = b$, "c" is called the **dividend, "a"** is called the **divisor** and "b" is called the **quotient** ; the quotient is a unique number. Also, "c" is frequently referred to as the **numerator (top)** and "a" the **denominator (bottom).** We can also write $b = \frac{c}{a} = c / a = c \div a$.

By the way, since 12 = 12*1 = 1*12 and 12 = 6*2 = 2*6, we also have

1. $\frac{12}{12} = 1$

2. $\frac{12}{1} = 12$
3. $\frac{12}{2} = 6$
4. $\frac{12}{6} = 2$

Additionally, we have

1. $\frac{25}{5} = 5 \text{ since } 25 = 5*5 = 5^2$
2. $\frac{32}{8} = 4 \;\&\; \frac{32}{4} = 8 \text{ since } 32 = 8*4 = 4*8$
3. $\frac{42}{6} = 7 \;\&\; \frac{42}{7} = 6 \text{ since } 42 = 6*7 = 7*6$

We can also easily include all of the integers if we are careful with the negative integers. Recall that if c = a*b so that if the factors "a" and "b" have the same sign, the quotient is positive and if the factors "a" and "b" have opposite signs, the quotient is negative:

1. $\frac{27}{-3} = -9 \text{ since } 27 = (-3)*(-9)$
2. $\frac{-36}{-3} = 12 \text{ since } -36 = (-3)*(12)$
3. $\frac{-16}{2} = -8 \text{ since } -16 = (2)*(-8)$

When there is a minus sign "-" , we can have three (3) choices where to place it:

1. In the numerator
2. In the denominator
3. Out in front

As an example, let's look at three (3) ways "- 4" can be written:

1. Since -12 = (3)*(- 4) , we have $-4 = \frac{-12}{3}$ (minus sign in numerator)
2. Since 12 = (-3)*(- 4), we have $-4 = \frac{12}{-3}$ (minus sign in denominator)

3. Since 12 = 3*4, we have $4 = \frac{12}{3}$ so that $-4 = -\frac{12}{3}$ (minus sign out in front)

Hence $-4 = -\frac{12}{3} = \frac{12}{-3} = \frac{-12}{3}$. In summary, if we have two (2) numbers "a & b"

and can form $\frac{a}{b}$, then $-\frac{a}{b} = \frac{a}{-b} = \frac{-a}{b}$. Note $b \neq 0$ as discussed below.

Note
$\frac{c}{a} \overset{Defined}{\equiv} b$ **when c = a*b** ; for now, both "a" and "b" are *positive* integers. **"c"** is **dividend** **"a" is divisor** **"b"** is **quotient ;** a unique number. This is called **closure**.

The number "0" plays a special role in the process of division. Consider $\frac{7}{0} = b = ?$. For any number we know, b*0 = 0, NOT 7! This means that

$\frac{7}{0}$ can NOT be defined. It means NOTHING! So if you see $\frac{c}{0}$, just say that it is undefined, does NOT exist, ... Question: When can we divide by "0"? Answer: NEVER, NEVER, NEVER, ... "0" cannot be in the denominator (bottom)! However, "0" can be in the numeration when it's not also in the denominator (top). In fact, $\frac{0}{7} = 0$ since 0 = 7*0. So, in general,

1. $\frac{c \neq 0}{0} =$ undefined (ignore $\frac{0}{0}$ if you see it)
2. $\frac{0}{a \neq 0} = 0$

Also, we have

1. $\frac{a}{a} = 1$ since $a = a*1 = 1*a$; $a \neq 0$
2. $\frac{a}{1} = a$ since $a = 1*a = a*1$

We have seen a property involving exponents and multiplication, namely $(a^n)^m = a^{n*m}$. Here is one more involving division with an example:

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}\ ;\ b \neq 0$$

Example: $\left(\frac{6}{3}\right)^2 = \frac{6^2}{3^2} = \frac{36}{9} = 4$ & $\left(\frac{6}{3}\right)^2 = 2^2 = 4$

Note
Three options: $-\frac{a}{b} = \frac{a}{-b} = \frac{-a}{b}\ ;\ b \neq 0$ $\frac{c \neq 0}{0} = \text{undefined}$ $\frac{0}{a \neq 0} = 0$ $\frac{a}{a} = 1; a \neq 0$ $\frac{a}{1} = a$ $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}\ ;\ b \neq 0$

Inequalities and the Operations: Addition, subtraction, multiplication, and division

We now present the **inequality rules** for **terms** (addition and subtraction) and then for **factors** (multiplication and division) with examples:

Terms: a, b, & c are numbers

1. If a < b, then a + c < b + c ; a – c < b – c
2. If a > b, then a + c > b + c ; a – c > b – c

Examples:

1. 3 < 5 ; 3 + 7 = 10 < 12 = 5 + 7
2. - 6 < 2 ; - 6 + 4 = -2 < 6 = 2 + 4
3. 7 > 2 ; 7 - 3 = 4 > - 1 = 2 – 3
4. - 5 > - 2 ; - 5 + 1 = - 4 > - 1 = - 2 + 1

Factors: a, b, & c are numbers … MUST be careful here!

Multiplication: $c \neq 0$

1. If $a \leq b \& c > 0$, then $a * c \leq b * c$ (Inequality: *same* direction)
2. If $a \leq b \& c < 0$, then $a * c \geq b * c$ (Inequality: *opposite* direction)
3. If $a \geq b \& c > 0$, then $a * c \geq b * c$ (Inequality: *same* direction)
4. If $a \geq b \& c < 0$, then $a * c \leq b * c$ (Inequality: *opposite* direction)

Examples:

1. $-2 \leq 3 \& c = 4$; $-8 \leq 12$
2. $4 \leq 7 \& c = -2$; $-8 \geq -14$ (Inequality: opposite direction)
3. $-5 \geq -7 \& c = 3$; $-15 \geq -21$
4. $4 \geq -3 \& c = -5$; $-20 \leq 15$ (Inequality: opposite direction)

Division: $c \neq 0$

1. If $a \leq b \& c > 0$, then $\frac{a}{c} \leq \frac{b}{c}$(Inequality: *same* direction)
2. If $a \leq b \& c < 0$, then $\frac{a}{c} \geq \frac{b}{c}$(Inequality: *opposite* direction)
3. If $a \geq b \& c > 0$, then $\frac{a}{c} \geq \frac{b}{c}$ (Inequality: *same* direction)
4. If $a \geq b \& c < 0$, then $\frac{a}{c} \leq \frac{b}{c}$ (Inequality: *opposite* direction)

Examples:

1. $12 \le 18 \;\&\; c = 2\;;\; \frac{12}{2} = 6 \le 9 = \frac{18}{2}$
2. $-12 \le -6 \;\&\; c = -3\;;\; \frac{-12}{-3} = 4 \ge 2 = \frac{-6}{-3}$ (Inequality: opposite direction)
3. $27 \ge -9 \;\&\; c = 3\;;\; \frac{27}{3} = 9 \ge -3 = \frac{-9}{3}$
4. $14 \ge 7 \;\&\; c = -7\;;\; \frac{14}{-7} = -2 \le -1 = \frac{7}{-7}$ (Inequality: opposite direction)

Applications of Multiplication with Positive Integers

Prime and Composite Numbers – Positive Integers (Natural Numbers)

We know that every positive integer, that is {1,2,3,4,5,...}, satisfies

a = a*1 = 1*a. For example, 5 = 5*1 and 12 = 12*1 (= 6*2 = 4*3).

Whereas "12" has several pairs of factors, "5" *only* has one pair. Any positive integer that *only* has itself and the number one (1) as its factors is called a **prime number**. Every other number is called a **composite number**. So "5" is a prime number and "12" is a composite number. The prime numbers between 2 and 100 are noted below. The number "1" is neither a prime or a composite number.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	94	96	97	98	99	100

Let's look at a few composite numbers and their factors:

1. $18 = 2*\overbrace{9}^{3*3} = 2*3*3$
2. $34 = 2*17$
3. $48 = 2*\overbrace{24}^{\overbrace{4}^{2*2}*\overbrace{6}^{2*3}} = 2*2*2*2*3$

4. $30 = 2 * \overbrace{15}^{3*5} = 2*3*5$
5. $21 = 3*7$
6. $99 = 3*\overbrace{33}^{3*11} = 3*3*11$

Note that each of these composite numbers *can* be factored with *only* prime factors. This is true in general and is an important fact in the World of Mathematics.

Multiple of a Positive Integer:

A **multiple** of a positive integer "a" is *any* positive integer that has "a" as a factor. Consider "12". We have

1. "24" is a multiple of 12 since 24 = 2*12
2. "96" is a multiple of "12" since 96 = 12*8
3. "19" is **not** a multiple "7" since 1*7 = 7, 2*7 = 14, 3*7 = 21, …

Also, 15 is a multiple of 3 and 5 ; 36 is a multiple of 2, 3, 4, 6, 9, 12, and 18. For a given positive integer "a", we can get as many multiples as we desire just by multiplying this number by 1, 2, 3, 4, ... The following table illustrates this idea:

	a 12	b 15
1	12	15
2	24	30
3	36	45
4	48	60
5	60	75
6	72	90
7	84	105
8	96	120
9	108	135

Given two (2) positive integers "a & b", we sometimes want/need to find their **least common multiple**, denoted **LCM(a , b)**. To find the **LCM**, we must

1. find the common multiples of the numbers
2. choose the *smallest* (least) common multiple

What is the LCD(12,15)? Looking at the table above, we see nine multiples for "12" and "15" and that the least multiple (in yellow) is "60":

LCD(12,15) = 60.

Here are a few examples:

LCM(12,18) = 36

	a 12	b 18
1	12	18
2	24	36
3	36	54
4	48	72
5	60	90
6	72	108
7	84	126
8	96	144
9	108	162

LCM(14,21) = 42

	a 14	b 21
1	14	21
2	28	42
3	42	63
4	56	84
5	70	105
6	84	126
7	98	147
8	112	168
9	126	189

LCM(5,7) = 35

	a 5	b 7
1	5	7
2	10	14
3	15	21
4	20	28
5	25	35
6	30	42
7	35	49
8	40	56
9	45	63

FYI: Given “a & b”, a*b is *always* a common multiple of “a & b” but is seldom the LCM. With “5” and “7”, it’s actually the LCM.

Greatest Common Factor:

Given “a & b”, we want to find its **greatest common factor**, denoted **GCF(a , b)**. First, we need to find all the factors the numbers have in common. Then, we just need to select the greatest (that is, largest) common factor. The GCF can the “1”. This is true if the two (2) numbers are prime:

$$5: \overbrace{1,5}^{\text{Factors}}$$

$$7: \overbrace{1,7}^{\text{Factors}}$$

The number “1” is both the common factor and the greatest common factor: GCF(5 , 7) = 1. Now, let’s consider the composite numbers “28” and “42”:

28:	**1**	**2**	**4**	**7**	**14**	**28**		
42:	**1**	**2**	**3**	**7**	**6**	**14**	**21**	**42**

The greatest common factor is 14 (in red): GCF(28 , 42) = 14

Two (2) more examples follow:

81:	**1**	**3**	**9**	**27**	**81**	
18:	**1**	**2**	**3**	**6**	**9**	**18**

We have GCF(81 , 18) = 9 (in red).

30:	**1**	**2**	**3**	**10**	**15**	**30**		
70:	**1**	**2**	**5**	**7**	**10**	**14**	**35**	**70**

We have GCF(30 , 70) = 10 (in red).

We can also find the GCF – Greatest Common Factor – of two (2) numbers using primes. Consider the numbers “36” and “48” and their prime factors:

36 =	**2**	*	**2**	*	**3**	*	**3**		
48 =	**2**	*	**2**	*	**2**	*	**2**	*	**3**

If we just multiply the common primes (in red), we get the GCF: GCF(36 , 48) = 12. Below are a few more examples:

50 =	**2**	*	**5**	*	**5**
42 =	**2**	*	**3**	*	**7**

We have GCF(50 , 42) = 2.

66 =	**2**	*	**3**	*	**11**		
84 =	**2**	*	**2**	*	**3**	*	**7**

We have GCF(66 , 84) = 6.

14 =	**2**	*	**7**
21 =	**3**	*	**7**

We have GCF(14 , 21) = 7.

Terms and Factors:

Term properties ("+" and "-") are *often different* from factor properties ("*" and "/"). For example, $(a*b)^n = a^n * b^n$. In particular,

$(2*5)^2 = 100 = 2^2 * 5^2$. But $(2+5)^2 = 7^2 = 49 \neq 29 = 4 + 25 = 2^2 + 5^2$

Order of Operations:

We *only* add, subtract, multiple, and divide two (2) numbers at a time: 7 + 9 = 16. But if we want to add three (3) numbers, say 7, 9, 13, officially we must group them: (7 + 9) + 13 = 16 + 9 = 29. For more complicated **numerical expressions** (numbers, +, -, *, /, exponents), we need to evaluate these expressions according to what is called the **order of operations**. These are the steps and order:

1. Eliminate grouping symbols ("( )", "[ ]", "{ }", …) – inside to outside.

   $$2\underbrace{\left(1-\overbrace{(3-5)}^{[Inside]}\right)}_{[\quad Outside \quad]}=2\underbrace{\left(1-(-2)\right)}_{[\quad Outside \quad]}=2(3)=6$$

2. Determine the numbers with powers – exponents.

   $3^2=9$

   $(-4)^3=-64$

3. Perform multiplication and division – left to right.

   $16\underset{Left}{/}2*\underset{Right}{4}=8*4=32$

   $16\underset{Left}{*}2/\underset{Right}{4}=32/4=8$

4. Perform addition and subtraction – left to right.

   $11\underset{Left}{-}4\underset{Right}{+}3=7+3=10$

   $11\underset{Left}{+}4\underset{Right}{-}3=15-3=12$

Let's consider several examples:

1. $3+2*(4-5)=3+2*(-1)=3-2=1$
2. $-2^3-(-2)^3=-8-(-8)=-8+8=0$
3. $6*2/3=12/3=4$
4. $6/2*3=3*3=9$
5. $3-5*(1-[4+3]+2^4)=3-5*(1-7+16)=3-5*10=3-50=-47$
6. $7+9-4=16-4=12$
7. $7-9+4=-2+4=2$; $7-9+4\neq 7-13=-6$
8. $2^3-3^2=8-9=-1$
9. $3*(2+5)=3*7=21$ or $3*(2+5)=3*2+3*5=6+15=21$

Note: There are many rules and there are frequently different paths to the same answer.