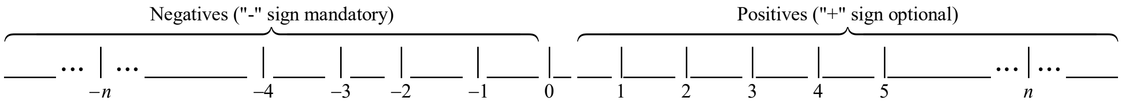
Real Number System

Rational and Irrational

(Types, Number Line, Equality, Inequality, Operations, Properties, ...)

Ra = Rational Numbers (aka, Ratio Numbers, Fractions, Q)

It’s time to add the “fractions”, also called the **ratio**nal numbers, to our current number line:



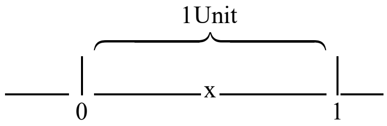
There are a lot of numbers to add! In general, **rational numbers** have the form $\frac{a}{b}$ where a , b are integers with $b \neq 0$:

$$\left\{ \frac{a}{b} \overset{\text{Such that}}{|} a, b \text{are integers with } \mathrm{b} \neq 0 \right\}$$

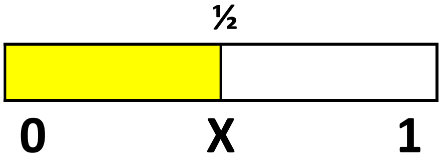
We have already seen that *integers* can be written in this form:

$$4 = \frac{4}{1} = \frac{12}{3} = \frac{24}{6} \ldots$$
$$3 = \frac{6}{2} = \frac{15}{5} = \ldots$$
$$-7 = \frac{7}{-1} = \frac{-7}{1} = \ldots$$

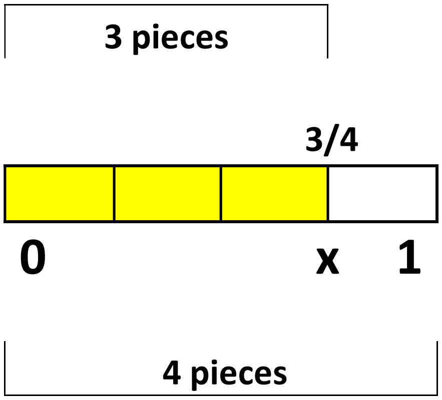
Now, we're ready to define rational numbers (fractions) that are NOT integers. Consider "0" and "1" on the number line:



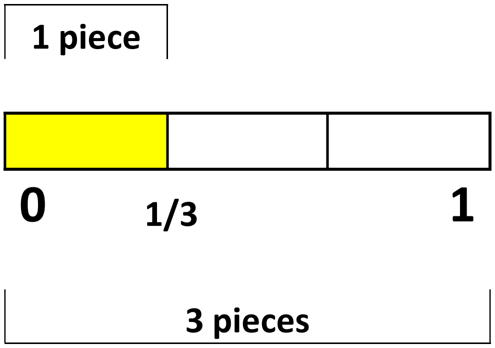
"x" marks the spot for a new number, written $\frac{1}{2}$. It is where the distance between "0" and "1", that is one (1) Unit, is divided into two (2) pieces:



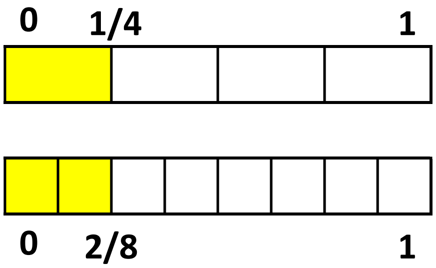
Next consider "0" to "1" being divided into four (4) equal pieces:



Now **x** represents the rational number $\frac{3}{4}$. The number $\frac{1}{3}$ is represented as follows:

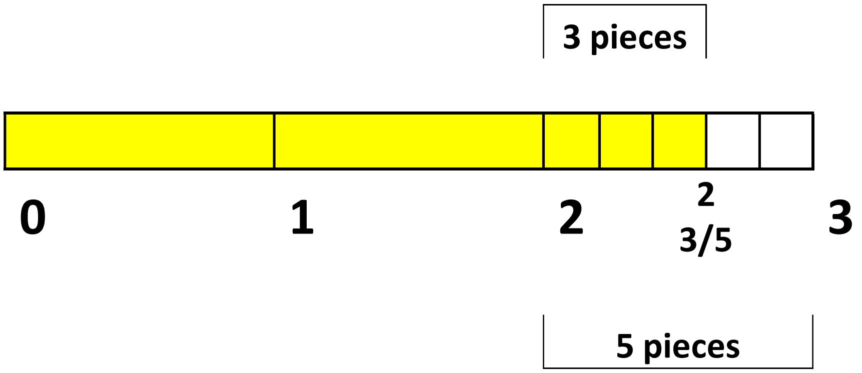


The next figure shows that $\frac{1}{4}$ and $\frac{2}{8}$ represent the same number; they are called **equivalent fractions** and are said to be equal: $\frac{1}{4}=\frac{2}{8}$.

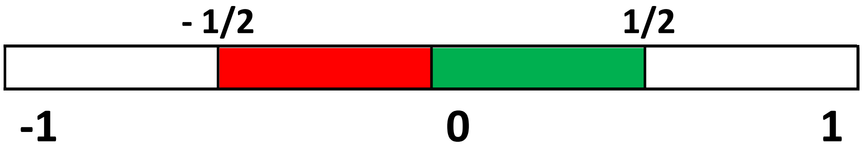


In a similar manner, we can show that $\frac{1}{4}=\frac{2}{8}=\frac{4}{16}=\frac{8}{32}=\ldots$. This means that there are an infinite number of ways to represent a fraction. *Equivalent* is just another way to say that the two (2) fractions are really the *same number* and are located at the same place on the number line.

We can easily represent fractions not between “0” and “1”. For example, if we want to represent $2+\frac{3}{5}$ (usually written $2\frac{3}{5}$ or 2 3/5), we could just construct the following figure:



Fractions can be negative too:

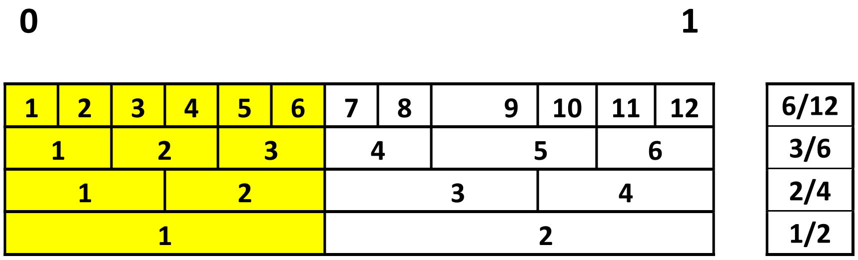


The distances are the same BUT opposite in direction, hence the "-" sign: $-\frac{1}{2}$.

The following diagram represents the fraction $-1\frac{2}{3}$

Note: $-1-\frac{2}{3}=-(1+\frac{2}{3})=-(1+2/3)=\begin{cases}-1\frac{2}{3}\\ -1\ \ 2/3\end{cases}$

Let’s do a little more work with equivalent fractions. Consider the following diagram:



Notice that $\frac{6}{12}=\frac{3}{6}=\frac{2}{4}=\frac{1}{2}$ and then notice that

1. $\frac{6}{12}=\frac{3}{6}$ & $6*6=12*3$
2. $\frac{3}{6}=\frac{2}{4}$ & $3*4=6*2$
3. $\frac{2}{4}=\frac{1}{2}$ & $2*2=4*1$

This example gives rise to the following truth:

$$\frac{a}{b}=\frac{c}{d}$$

is equivalent to

$$a*d=b*c$$

Note: Going from $\frac{a}{b}=\frac{c}{d}$ to $a*d=b*c$ is called **cross multiplication**.

Example: Which of the following pairs of fractions are equivalent?

a. $\frac{3}{7} = \frac{4}{5}$; $3*5 = 15 \neq 28 = 7*4$; Not equivalent
b. $\frac{4}{9} = \frac{12}{27}$; $4*27 = 108 = 9*12$; Equivalent
c. $\frac{1}{5} = \frac{2}{7}$; $5*2 = 10 \neq 7 = 1*7$; Not equivalent
d. $\frac{4}{7} = \frac{12}{21}$; $4*21 = 84 = 7*12$; Equivalent

Given a fraction, say $\frac{3}{8}$, we can easily find as many equivalent fractions $\frac{c}{d}$ as we want. If we form $\frac{c}{d} = \frac{3*n}{8*n}$ for $n = 1,2,3,4,\ldots$, we'll get equivalent fractions since 3*(8*n) = 8*(3*n):

1. $\frac{3}{8}$; n = 1
2. $\frac{6}{16}$; n = 2
3. $\frac{9}{24}$; n = 3
4. $\frac{12}{32}$; n = 4
5. …

We now know that two (2) fractions $\frac{a}{b} = \frac{c}{d}$ are equivalent (also said to be equal) when and only when a*d = b*c. If they are NOT equal $(\neq)$, can we determine if $\frac{a}{b} < \frac{c}{d}$ or $\frac{a}{b} > \frac{c}{d}$? The answer is YES! Let's look at a couple of examples:

1. $\frac{3}{8}\begin{Bmatrix}?\\<\\>\end{Bmatrix}\frac{5}{16}$; $3*16 = 48 > 40 = 8*5$; $a*d > b*c$

We see from above that $\frac{3}{8} = \frac{6}{16}$. Since 6 > 5, $\frac{3}{8} = \frac{6}{16} > \frac{5}{16}$ so that $\frac{3}{8} > \frac{5}{16}$. The *key* is to make the denominators of the fractions the same!

2. $\frac{2}{7}\left\{\begin{matrix} ? \\ < \\ > \end{matrix}\right\}\frac{3}{5}$; $2*5 = 10 < 21 = 7*3$; $a*d < b*c$

We first get equivalent fractions with the same denominator:

a. Using the denominator “5” in the second fraction, we get

$$\frac{2}{7} = \frac{2*5}{7*5} = \frac{10}{35}$$

b. Using the denominator “7” in the first fraction, we get

$$\frac{3}{5} = \frac{3*7}{5*7} = \frac{21}{35}$$

Since 21 > 10, we obtain $\frac{2}{7} < \frac{3}{5}$

Looking at the relationship between a*d and b*c, we find the following truth:

Assuming $a \neq 0,\ b > 0, c \neq 0, \& d > 0$, then

1. $\frac{a}{b} < \frac{c}{d}$ when $a*d < b*c$
2. $\frac{a}{b} = \frac{c}{d}$ when $a*d = b*c$
3. $\frac{a}{b} > \frac{c}{d}$ when $a*d > b*c$

A few examples follow:

a	b	a/b	c	d	c/d	a*d ? b*c	a/b ? c/d
-1	2	-1/2	-2	3	-2/3	>	>
3	6	3/6	1	2	1/2	=	=
-3	7	-3/7	2	5	2/5	<	<
4	9	4/9	5	7	5/7	<	<
-8	11	-8/11	-9	13	-9/13	<	<

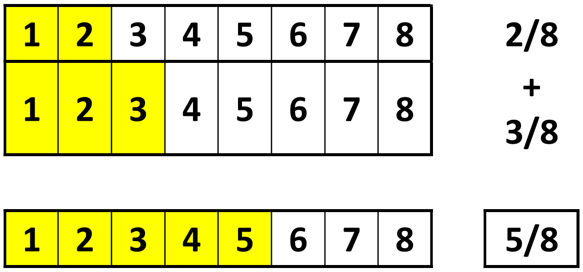
Operations on numbers: New numbers from given numbers

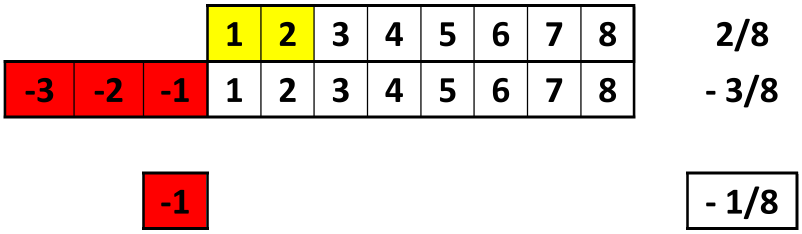
Addition/Subtraction of Rational Numbers – Fractions: Given two (2) fractions $\frac{a}{b}$ and $\frac{c}{b}$ with the *same* denominators, to get their **sum**, we just add their numerators:

$$\frac{a}{b} \pm \frac{c}{b} = \frac{a \pm c}{b}$$

Note: “b” is called the **common denominator**. For example,

$$\frac{2}{8} + \frac{3}{8} = \frac{2+3}{8} = \frac{5}{8}$$
$$\frac{2}{8} - \frac{3}{8} = \frac{2-3}{8} = \frac{-1}{8} \left(\text{also written as } -\frac{1}{8} \right)$$





			1	2	3	4	5	6	7	8	2/8
-3	-2	-1	1	2	3	4	5	6	7	8	- 3/8
		-1									- 1/8

Given two (2) fractions with *different* denominators, we have to make their denominators the same using equivalent fractions before we can add or subtract. We can use the following formula, but the *key* is to make their denominators the same:

$$\frac{a}{b} \pm \frac{c}{d} = \frac{a*d}{b*d} \pm \frac{b*c}{b*d} = \frac{a*d \pm b*c}{b*d}$$

Using this formula, we obtain

$$\frac{2}{5} + \frac{3}{4} = \frac{2*4}{5*4} + \frac{3*5}{4*5} = \frac{8+15}{20} = \frac{23}{20}$$
$$\frac{2}{5} - \frac{3}{4} = \frac{2*4}{5*4} - \frac{3*5}{4*5} = \frac{8-15}{20} = -\frac{7}{20}$$

Notice, we traded $\frac{2}{5}$ and $\frac{3}{4}$ for equivalent fractions that have the *same* denominator:

$$\frac{2}{5} = \frac{8}{20}$$
$$\frac{3}{4} = \frac{15}{20}$$

If we look at the multiples of “4” and “5”, we see that “20” is their **least common multiple**: LCM(4,5) = 20

LCM(4,5) = 20

	a 4	b 5
1	4	5
2	8	10
3	12	15
4	16	20
5	20	25
6	24	30
7	28	35
8	32	40
9	36	45

The product b*d will *always* be a common denominator but using the LCM(b, d) will give us the **least common denominator**, denoted LCD(b, d). Hence, we’ll have smaller numbers to work with! For example, to find $\frac{11}{12}+\frac{17}{18}$, we could use 12*18 = 216 as a *common denominator* but, if we find the LCM(12, 18) = , we’ll get the *least common denominator*:

LCM(12, 18) = 36

	a 12	b 18
1	12	18
2	24	36
3	36	54
4	48	72
5	60	90
6	72	108
7	84	126
8	96	144
9	108	162

Recall that we can also find the least common multiple using prime factors:

12 = 2 * 2 * 3

18 = 2 * 3 * 3

so that LCD(12 , 18) = 2*2*3*3 = 36

Using the LCD, we obtain

$$\frac{11}{12}+\frac{17}{18}=\frac{11*3}{12*3}+\frac{17*2}{18*2}=\frac{33}{36}+\frac{34}{36}=\frac{67}{36}$$

Note that using the LCD gives us a smart way to find equivalent fractions for $\frac{11}{12}\text{and}\frac{17}{18}$. Let's find $\frac{3}{14}-\frac{1}{10}$. Using prime factors to find the LCD(14,10), we have

14 = 2 * 7

10 = 2 * 5

so that LCD(14, 10) = 70. Now,

$$\frac{3}{14}-\frac{1}{10}=\frac{3*5}{14*5}-\frac{1*7}{10*7}=\frac{15-7}{70}$$
$$=\frac{8}{70}=\frac{2*4}{2*35}=\frac{4}{35}\ (\text{in simplified form})$$

Mixed fractions:

The number, for example, 3 2/5 is called a **mixed fraction.** We can write it in its **fractional form** using the least common denomination process:

$$3\,\frac{2}{5}=3+\frac{2}{5}=\frac{3}{1}+\frac{2}{5}=\frac{3*5}{1*5}+\frac{2}{5}=\frac{17}{5}$$

Also

$$-2\,\frac{3}{7}=-\left(2+\frac{3}{7}\right)=-\left(\frac{2}{1}+\frac{3}{7}\right)=-\left(\frac{2*7}{1*7}+\frac{3}{7}\right)$$
$$=-\frac{17}{7}$$

If we look at this process, we can obtain a quicker way to obtain fractional form: $a\,\frac{c}{d}=\frac{a*d+c}{d}$

By the way:

1. If the absolute value of the fraction is less than one, it’s called a **proper fraction**.
2. If the absolute value of the fraction is greater than one, it’s called an **improper fraction**.

Multiplication of Rational Numbers – Fractions: Given two (2) fractions $\frac{a}{b}$ and $\frac{c}{d}$; $b,d \neq 0$, the **product** denoted $\frac{a}{b}*\frac{c}{d}$, is defined by

$$\frac{a}{b}*\frac{c}{d} \equiv \frac{a*c}{b*d}.$$

Note that we just multiple the numerators and then multiple the denominators. Addition, for example, does NOT work that way:

$$\frac{a}{b}+\frac{c}{d} \neq \frac{a+c}{b+d}$$

Some examples of multiplication follow:

1. $\frac{3}{5}*\frac{2}{7}=\frac{3*2}{5*7}=\frac{6}{35}$
2. $\frac{3}{11}*\frac{2}{3}=\frac{3*2}{11*3}=\frac{6}{33}=\frac{2}{11}$ (we always require simplified form)
3. $\frac{2}{5}*3=\frac{2}{5}*\frac{3}{1}=\frac{6}{5}$
4. $2\,\frac{1}{3}*\frac{4}{5}=\frac{7}{3}*\frac{4}{5}=\frac{28}{15}=\frac{15}{15}+\frac{13}{15}=1+\frac{13}{15}=1\,\frac{13}{15}$
5. $\frac{2}{3}*\frac{3}{5}+\frac{1}{4}=\frac{2*3}{3*5}+\frac{1}{4}=\frac{6}{15}+\frac{1}{4}=\frac{2}{5}+\frac{1}{4}=\frac{2*4}{5*4}+\frac{1*5}{4*5}=\frac{8}{20}+\frac{5}{20}=\frac{13}{20}$

Division of Rational Numbers – Fractions: Given two (2) fractions $\frac{a}{b}$ and $\frac{c}{d}$; $b,c,d \neq 0$, the **quotient**, denoted $\frac{a}{b}\,/\,\frac{c}{d}$ or $\frac{a}{b} \div \frac{c}{d}$ or $\frac{\frac{a}{b}}{\frac{c}{d}}$, is defined by

$$\frac{a}{b}\,/\,\frac{c}{d} = \frac{a}{b} \div \frac{c}{d} = \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b}*\frac{d}{c}.$$

Before we give examples, we introduce the **reciprocal of a number** $a \neq 0$. It's simply $\frac{1}{a}$:

1. $5 = \frac{5}{1}$: Reciprocal $= \frac{1}{5}$
2. $\frac{3}{7}$: Reciprocal $= \frac{1}{\frac{3}{7}} = \frac{\frac{1}{1}}{\frac{3}{7}} = \frac{1}{1} * \frac{7}{3} = \frac{7}{3}$
3. $-\frac{2}{5}$: Reciprocal $= \frac{1}{-\frac{2}{5}} = \frac{\frac{1}{1}}{-\frac{2}{5}} = \frac{1}{1} * \left(-\frac{5}{2}\right) = -\frac{5}{2}$
4. $\frac{c}{d}$: Reciprocal $= \frac{1}{\frac{c}{d}} = \frac{\frac{1}{1}}{\frac{c}{d}} = \frac{1}{1} * \frac{d}{c} = \frac{d}{c}$

Note that $a * \frac{1}{a} = \frac{a}{1} * \frac{1}{a} = \frac{a}{a} = 1$

With the reciprocal in mind the quotient of $\frac{a}{b}$ and $\frac{c}{d}$; $b, c, d \neq 0$ is the numerator fraction multiplied by the reciprocal of the denominator fraction:

$$\frac{a}{b} / \frac{c}{d} = \frac{a}{b} \div \frac{c}{d} = \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} * \frac{d}{c}$$

Now for some examples:

1. $\frac{4}{3}$ divided by $3 = \frac{3}{1}$ equals $\frac{4}{3} * \frac{1}{3} = \frac{4}{9}$
2. $-\frac{3}{7} \div \frac{2}{5} = -\frac{3}{7} * \frac{5}{2} = -\frac{3*5}{7*2} = -\frac{15}{14} = -1\frac{1}{14}$
3. $-\frac{1}{6} / \frac{5}{2} = -\frac{1}{6} * \frac{2}{5} = -\frac{1*2}{6*5} = -\frac{2}{30} = -\frac{1}{15}$ (in simplified form)

4. $-\frac{-\frac{3}{7}}{\frac{13}{14}} = \left(-\frac{3}{7}\right)*\left(-\frac{14}{13}\right) = \frac{3*14}{7*13} = \frac{3*2*7}{7*13} = \frac{6}{13}$

Exponentiation:

NEVER allowing division by zero and always excluding 0^0, we can update our **power number** definitions to include rational numbers:

$$\text{Base}^{\text{Exponent}} = \left(\frac{a}{b}\right)^n \overset{defined}{=} \overbrace{\left(\frac{a}{b}\right)*\left(\frac{a}{b}\right)*\left(\frac{a}{b}\right)*\ldots*\left(\frac{a}{b}\right)}^{\text{n times}} = \frac{a^n}{b^n}$$

where “a” and “b” are integers and “n” is a positive integer or zero. Now we can also allow the “Exponent” to be negative; assuming that “n” is a positive integer, we define

$$\text{Base}^{\text{Exponent}} = \left(\frac{a}{b}\right)^{-n} \overset{defined}{=} \left(\frac{b}{a}\right)^n$$

If b =1, we have $a^{-n} = \frac{1}{a^n}\ ;\ \frac{1}{a^{-n}} = a^n$

Several examples follow:

1. $\left(\frac{3}{4}\right)^2 = \left(\frac{3}{4}\right)*\left(\frac{3}{4}\right) = \frac{9}{16} = \frac{3^2}{4^2}$
2. $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$

3. $\left(\frac{5}{2}\right)^{-1} = \left(\frac{2}{5}\right)^{1} = \frac{2}{5}$

4. $\frac{1}{3^{-3}} = 3^{3} = 27$

5. $\left(\frac{2}{3}\right)^{-4} = \left(\frac{3}{2}\right)^{4} = \frac{3^4}{2^4} = \frac{81}{16}$

6. $\left(\frac{4}{5}\right)^{3} = \frac{4^3}{5^3} = \frac{64}{125}$

Note

$$\left(\frac{a}{b}\right)^{n} \overset{defined}{=} \frac{a^n}{b^n}$$

$$\left(\frac{a}{b}\right)^{-n} \overset{defined}{=} \left(\frac{b}{a}\right)^{n}$$

Decimal Representation of Fractions:

We have seen that the integers can be written in **fractional form:**

$$5 = \frac{5}{1}\,;\, 17 = \frac{17}{1}\,;\, -11 = -\frac{11}{1} = \ldots$$

They are easily written in **decimal form**:

$$5. = \frac{5.}{1}\,;\, 17. = \frac{17.}{1}\,;\, -11. = -\frac{11.}{1} = \ldots$$

where the "." at the right-hand side and bottom of the integer is called the **decimal point**.

Note that $4\cdot 8=32$ but 4.8 does NOT mean multiplication.

Non-integer fractions also have a **decimal representation** (**decimal form**) which we now discuss. First consider an integer such as 2734 (= 2734.). We can write it as

2734 = 2*1000 + 7*100 + 3*10 + 4

In chart form, we have

Sign	Thousands (*1000)	Hundreds (*100)	Tens (*10)	Units	Decimal Point
+	2	7	3	4	.

Here are a few more examples using integers:

Sign	10 Thousands (*10000)	Thousands (*1000)	Hundreds (*100)	Tens (*10)	Units	Decimal Point
+		2	7	3	4	.
-	3	0	1	2	5	.
+				7	3	.
-		4	8	2	4	.
+	6	8	7	5	0	.

By the way, we usually use a comma “,” to separate “thousands”:

243765887. = 243,765,887.

When we deal with fractions, we frequently need tenths, hundredths, thousandths, ... : $\frac{1}{10}$; $\frac{1}{100}$; $\frac{1}{1000}$; $\frac{1}{10000}$; ... : Here is a chart for the decimal number 53724.569:

Sign	10 Thousands (*10000)	Thousands (*1000)	Hundreds (*100)	Tens (*10)	Units	Decimal Point	Tenths (*1/10)	Hundredths (*1/100)	Thousands (*1/1000)
+	5	3	7	2	4	.	5	6	9

Thus 53724.569 = 5*10000 + 3*1000 + 7*100 + 2*10 + 4 + 5*1/10

+ 6*1/100 + 9*1/1000

$$= \frac{53724569.}{1000}$$ (Common denominator = 1000)

Here are some decimal numbers with their fractional equivalents:

1. $2.34 = \frac{2}{1} + \frac{3}{10} + \frac{4}{100} = \frac{2*100}{1*100} + \frac{3*10}{10*10} + \frac{4}{100} = \frac{200+30+4}{100} = \frac{234.}{100}$
2. $-0.2 = -\frac{2.}{10}$
3. $0.45 = \frac{4}{10} + \frac{5}{100} = \frac{4*10}{10*10} + \frac{5}{100} = \frac{45.}{100}$

Multiplication and Division by Powers of 10:

Multiplication:

Notice that

1. $5.000 * 10^1 = 50.000$
2. $5.000 * 10^2 = 5.000 * 100 = 500.000$
3. $5.000 * 10^3 = 5.000 * 1000 = 5000.000$
4. …

What the examples are telling us to do is to move the decimal point to the "***right***" the same number of places as the power (1,2,3,...).

Division:

Notice that

1. $\frac{5.000}{10^1} = 0.5000 \text{ (since } 0.5000 * 10 = 5.000\text{)}$

2. $\frac{5.000}{10^2}=\frac{5.000}{100}=0.05000\left(\text{since } 0.05000 * 100 = 5.000\right)$
3. $\frac{5.000}{10^3}=\frac{5.000}{1000}=0.005000\left(\text{since } 0.005000 * 1000 = 5.000\right)$
4. …

What the examples are telling us to do is to move the decimal point to the "***left***" the same number of places as the power (1,2,3,...).

Caution: We must be careful where the decimal point is and what operations we are asked to perform:

$$4 \bullet 3 + 4^3 + 4.3 = 12 + 64 + 4.3 = 80.3$$

Decimal form *to* Fraction form and Fraction form *to* Decimal form:

We have seen how to go from decimal form to fraction form:

$$3.4 = 3 + \frac{4}{10} = \frac{3}{1} + \frac{4}{10} = \frac{3*10}{1*10} + \frac{4}{10} = \frac{34}{10} = \frac{17*2}{5*2} = \frac{17}{5}$$

Going from fractional form to decimal form involves division. We have seen division before:

$$\frac{12}{3} = 4$$

since 12 = 3*4. Another way to obtain this is to use **long division** which has the form:

$$\begin{array}{l} \qquad\;\;\text{Quotient} \\ \text{Divisor}\overline{\smash{)}\text{Dividend}} \\ \qquad\;\; . \\ \qquad\;\; . \\ \qquad\;\; \underline{\ldots} \\ \qquad\quad \text{Remainder} \end{array} \qquad ; \qquad \frac{\text{Dividend}}{\text{Divisor}} = \text{Quotient}$$

We have

$$\begin{array}{l} \qquad\quad\;\; 3 \\ \text{Times } 4\overline{\smash{)}12} \\ \qquad\quad\; \underline{12}\ \text{ Subtract} \\ \qquad\quad\;\; 0\ \text{ Remainder} \end{array}$$

Thus, $\frac{12}{3} = 4$.

However, with $\frac{17}{5}$ there is NOT an integer “a” such that 17 = 5*a:

$$\begin{array}{l} 5*1 = 5 \;\; \text{to small} \\ 5*2 = 10 \;\; \text{to small} \\ 5*3 = 15 \;\; \text{to small} \\ 5*4 = 20 \;\; \text{to big} \end{array}$$

The number a = 3 is “close”:

17. = 17 = 5*3 + 2

We interpret this as follows:

$$\begin{array}{rl} & \;\;3. \\ \text{Times } 5 & \!\!\big)\overline{17.0} \\ & \;\underline{15}\ \text{Subtract} \\ & \;\;\,2 \end{array}$$

$$\begin{array}{rl} & \;\;3.4 \\ \text{Times } 5 & \!\!\big)\overline{17.0} \\ & \;\underline{15} \\ & \;\;\,20\ \ \text{Bring down a "0"} \\ & \;\underline{20}\ \ \text{Subtract} \\ & \;\;\;0\ \text{Remainder} \end{array}$$

Thus, $\frac{17}{5} = 3.4$ since 17. = 5*3.4. To check this, note that

$$3.4 = 3 + 0.4 = \frac{3}{1} + \frac{4}{10} = \frac{3*10}{1*10} + \frac{4}{10} = \frac{30}{10} + \frac{4}{10} = \frac{34}{10} = \frac{17}{5}$$

Important: Note how the decimal points above were “aligned”.

Let’s find the decimal form of 29.61 divided by 7, that is $\frac{29.61}{7}$. We have

$$\begin{array}{rl} & \;\;4. \\ \text{Times } 7 & \!\!\big)\overline{29.61} \\ & \;\underline{28}\ \text{Subtract} \\ & \;\;1 \end{array}$$

$$\begin{array}{l} \qquad\quad 4.2 \\ \text{Times } 7\overline{)29.61} \\ \qquad\ \underline{28} \\ \qquad\quad 16 \ \ \text{Bring down the "6"} \\ \qquad\quad \underline{14} \ \ \text{Subtract} \\ \qquad\quad\ \ 2 \end{array}$$

$$\begin{array}{l} \qquad\quad 4.23 \\ \text{Times } 7\overline{)29.61} \\ \qquad\ \underline{28} \\ \qquad\quad 16 \\ \qquad\quad \underline{14} \\ \qquad\qquad 21 \ \text{Bring down the "1"} \\ \qquad\qquad \underline{21} \ \ \text{Subtract} \\ \qquad\qquad\ \ 0 \ \ \text{Remainder} \end{array}$$

Thus, $\frac{29.61}{7} = 4.23$. Let's check this one:

$$7 * 4.23 = 7\left(4 + \frac{2}{10} + \frac{3}{100}\right) = 28 + \frac{14}{10} + \frac{21}{100} = 28 + \left(\frac{10+4}{10}\right) + \left(\frac{20+1}{100}\right)$$

$$= 28 + \left(1 + \frac{4}{10}\right) + \left(\frac{2}{10} + \frac{1}{100}\right) = 29.61$$

Here are a few more examples:

1. $\frac{306}{5} = ?$

$$\begin{array}{l} \qquad\qquad\ \ 6 \\ \text{Times } 5\overline{)306.0} \\ \qquad\qquad \underline{30} \ \ \text{Subtract} \\ \qquad\qquad\quad 6 \ \ \text{Bring down the "6"} \end{array}$$

$$\begin{array}{r} 61 \\ \text{Times } 5\overline{)306.0} \end{array}$$

$\underline{30}$

6

$\underline{5}$ Subtract

10 Bring down the "0"

$$\begin{array}{r} 61.2 \\ \text{Times } 5\overline{)306.0} \end{array}$$

$\underline{30}$

6

$\underline{5}$

10

$\underline{10}$ Subtract

0 Remainder

2. $\frac{825}{3} = ?$ We will NOT put all of the reminders in from now on.

$$\begin{array}{r} 275.2 \\ 3\overline{)825.6} \end{array}$$

$\underline{6}$

22

$\underline{21}$

15

$\underline{15}$

6

$\underline{6}$

0

Notice that every quotient so far terminated, that is, it ended in zeroes:

- 4.00…
- 3.400…
- 4.2300…
- 61.200…
- 275.200…

Fractions may also have a repeating patten other than zeroes which we now explore. Consider the fraction $\frac{2}{3}$. Using long division, we obtain

$$\begin{array}{l}
\qquad\qquad\ \ \overline{0.666...} \\
\text{Times } 3\big)\ 2.0000 \\
\qquad\qquad\ \ \underline{18} \\
\qquad\qquad\quad\ 20 \\
\qquad\qquad\quad\ \underline{18} \\
\qquad\qquad\qquad 20 \\
\qquad\qquad\qquad \underline{18} \\
\qquad\qquad\qquad\ \ 20 \\
\qquad\qquad\qquad\ \ \underline{18} \\
\qquad\qquad\qquad\quad 20 \\
\qquad\qquad\qquad\quad ...
\end{array}$$

Notice that the pattern

$$\begin{array}{l}
20 \\
\underline{18} \\
\ \ 20
\end{array}$$

repeats for ever so we write $\frac{2}{3} = 0.66666... \overset{\text{OR}}{=} 0.6666\,\overline{6}$. The three dots

(...) or the bar (¯) are the notations meaning that the pattern continues for ever. Getting the fraction from the decimal form is a little more involved. Since the decimal repeats after only one digit (1), we do the following calculation, letting “d” represent the decimal: $d = 0.6666\overline{6}$

$$\begin{array}{l} 10d = 6.6666\overline{6} \quad \text{Recall that multiplication by 10 moves the decimal point one} \\ \qquad\qquad\qquad\quad \text{place to the right} \\ \underline{\quad d = 0.6666\overline{6}} \quad \text{Subtract} \\ 9d = 6 \\ d = \frac{6}{9} = \frac{2}{3} \end{array}$$

What fraction does $d = 2.141414\overline{14}$ represent? Since the decimal repeats after two (2) digits, we multiple by 100:

$$\begin{array}{l} 100d = 214.1414\overline{14} \quad \text{Recall that multiplication by 100 moves the decimal point two} \\ \qquad\qquad\qquad\qquad \text{places to the right} \\ \underline{\quad d = 2.1414\overline{14}} \quad \text{Subtract} \\ 99d = 212 \\ d = \frac{212}{99} \end{array}$$

Irr = Irrational Numbers (Non-rational Numbers, Non-fractions)

After the rational numbers (fractions) were constructed and placed on the number line, the "Math Gurus" thought that every point on the number line could be represented by a fraction. They were wrong! A man named Pythagoras came along and proved that there were an infinite number of "other" numbers that were NOT fractions. These other numbers were called **irrational numbers**. In "real life" applications, these irrational numbers play an important role.

We have seen that **rational numbers**, that is fractions, have decimal expansions that

1. terminate: $\frac{1}{2} = 0.5 = 0.500000000...$

or

2. repeat: $\frac{2}{3} = 0.6666\overline{6} = 0.66666...$

The **irrational numbers** also have decimal expansions, but they do NOT terminate nor repeat:

For example,

$\pi = 3.141592653589879$

$e = 2.71828182845904...$

5.101001000100001000001...

$\sqrt{2} = 1.41421356237309...$ (called the **square root of 2**)

Square roots will be discussed below.

When we operate (+,-,*,/,...) with rational numbers (fractions), we get *exact* answers ; when we operate with irrational numbers, we only get *approximate* answers.

We have previously discussed **power numbers** of the form

$$\left(\frac{a}{b}\right)^{n}$$

where " $\frac{a}{b}$ " is a rational number and "n" is an integer (0^{0} is NOT allowed).

Now, we consider **power numbers** of the form

$$\left(\frac{a}{b}\right)^{r}$$

where

1. b = 1
2. "r" is the fraction 1/2 or 1/3

Let us consider r = 1/2 first: $a^{1/2}$ with $a \geq 0$. Note that $a^{1/2}$ is frequently written as $a^{1/2} = \sqrt[2]{a} = \sqrt{a}$ and is called the **square root of** "a". The $\sqrt{a}$ represents a number "c" such that $c^{2} = a$. There are some *nice* square roots:

1. $\sqrt{0} = 0$ since 0*0 = 0
2. $\sqrt{1} = 1$ since 1*1 = 1
3. $\sqrt{4} = 2$ since 2*2 = 4
4. $\sqrt{9} = 3$ since 3*3 = 9
5. $\sqrt{16} = 4$ since 4*4 = 16
6. $\sqrt{25} = 5$ since 5*5 = 25

7. $\sqrt{36} = 6$ since 6*6 = 36
8. …

Notice that for every positive integer "a"

$$\sqrt{a^2} = a$$

The reason we have postponed talking about square roots until now is that they frequently result in irrational numbers, that is, numbers whose decimal expansions (representations, forms) do not terminate or repeat. In the past, tables were constructed that contained decimal approximations of square roots that resulted in irrational numbers. The good news is that now we can use a "scientific calculator" to get approximate values easily – we just use the "$\sqrt{\ }$" key. For example,

$$\sqrt{2} \approx 1.414214$$

to six (6) decimals. A few more square root approximations are given below:

$$\sqrt{3} \approx 1.732051$$

$$\sqrt{5} \approx 2.236068$$

$$\sqrt{17} \approx 4.123106$$

$$\sqrt{23} \approx 4.795832$$

Now consider r = 1/3: $a^{1/3}$ where "a" is now any integer. Note that $a^{1/3}$ is frequently written as $a^{1/3} = \sqrt[3]{a}$ and is called the **cube root of** "a". The $\sqrt[3]{a}$ represents a number "c" such that $c^3 = a$. Here are some *nice* cube roots:

1. $\sqrt[3]{8} = 2$ since $2^3 = 8$
2. $\sqrt[3]{-8} = -2$ since $(-2)^3 = -8$
3. $\sqrt[3]{-125} = -5$ since $(-5)^3 = -125$

Note that whereas $\sqrt{a} \geq 0$, $\sqrt[3]{a}$ can be negative, zero, or positive.

Not so nice ones resulting in irrational numbers include

1. $\sqrt[3]{11} = 2.223980$
2. $\sqrt[3]{-19} = -2.668402$
3. $\sqrt[3]{41} = 3.448217$

We can also find square and cube roots of "$\frac{a}{b}$": $\sqrt{\frac{a}{b}}, \left(\frac{a}{b} \geq 0\right); \sqrt[3]{\frac{a}{b}}$

1. $\sqrt[3]{\frac{64}{27}} = \frac{4}{3}$ since $\left(\frac{4}{3}\right)^3 = \frac{64}{27}$
2. $\sqrt[3]{\frac{1}{8}} = \frac{1}{2}$ since $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$
3. $\sqrt[3]{\frac{5}{7}} \approx 0.893904$
4. $\sqrt[3]{\frac{23}{13}} \approx 1.209469$

Actually, $\text{Base}^{\text{Exponent}}$ can frequently be defined when both the Base and Exp are rational or irrational numbers. For now, we will just use the "^" key on a scientific calculator to obtain these values.