Continuity

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In the finite two-sided limit material, the limit of several types of functions at a value in their domain was *just* the value of the function at this value – nice … Polynomial, rational, exponential, logarithmic, trigonometric, … functions were among these types of functions. Geometrically, this means that there is **NO BREAK** in the graph of the function at this domain value. No holes or other problems in the graph at this value! The function is said to be **continuous** at this **x** value. We want to determine all these values, denoted **Continuity f** = **CONT f**. At the other points, we want to determine the type of break – **discontinuity** – and how it affects the graph. There is a three (3) step process in verifying the continuity of a function at an **x** value:

Definition: A function $\mathbf{f}$ is **continuous** at $\mathbf{x}_0 \in \mathbb{R}$ if

1. $\mathbf{x}_0 \in \mathbf{Dom\ f}$
2. $\lim\limits_{\mathbf{x} \to \mathbf{x}_0} \mathbf{f}(\mathbf{x}) \in \mathbb{R}$
3. $\lim\limits_{\mathbf{x} \to \mathbf{x}_0} \mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x}_0)$

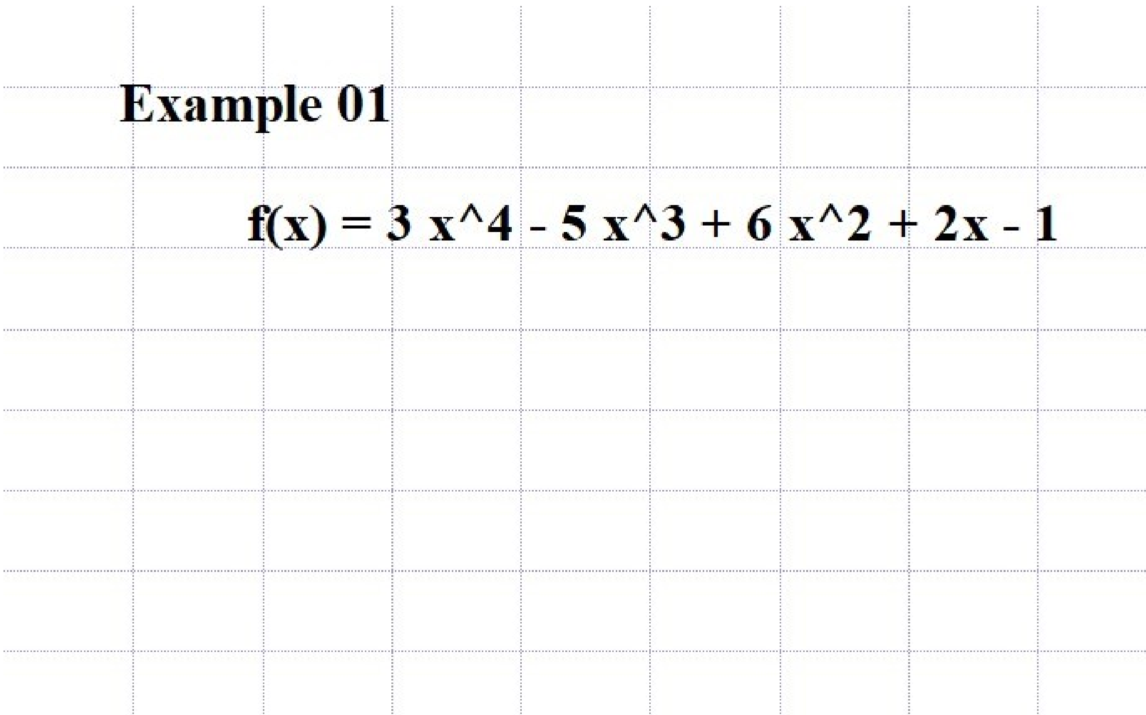
We will apply this definition as appropriate but appeal to the many theorems that follow from the definition. No proofs of these theorems will be given at this time, BUT make sure you know how to apply them!

Example 01: Given $\mathbf{f}(\mathbf{x}) = 3\mathbf{x}^4 - 5\mathbf{x}^3 + 6\mathbf{x}^2 + 2\mathbf{x} - 1$, find $\textbf{Cont f}$.

Solution: Here, we just appeal to one of the continuity theorems that states that polynomial functions are *always* continuous:

$$\textbf{Cont f} = \textbf{Dom f} = \mathbb{R}_{\mathbf{x}}$$

Note that there are no breaks in the following graph.



Example 02: Given $\mathbf{f}(\mathbf{x}) = \dfrac{\mathbf{x}^3 + 5\mathbf{x}^2 - 24\mathbf{x}}{\mathbf{x}^2 - 4\mathbf{x} - 12}$, find **Cont f**

Solution: Here, we just appeal to one of the continuity theorems that states that rational functions are *always* continuous at **x** values in their domains:

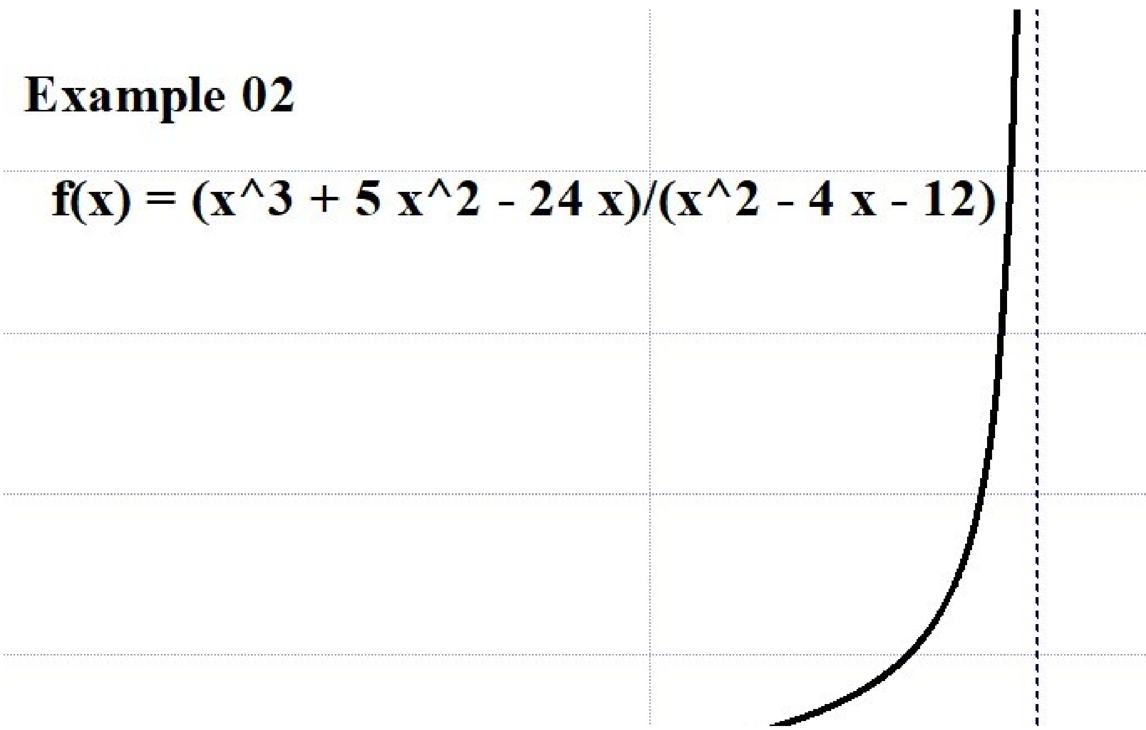
$$\textbf{Cont f} = \textbf{Dom f}$$

We must determine the real numbers NOT in the domain of **f**:

$$\text{Set } \mathbf{x}^2 - 4\mathbf{x} - 12 \overset{\textbf{SET}}{=} 0 \Rightarrow (\mathbf{x} - 6)\,(\mathbf{x} + 2) = 0 \Rightarrow \mathbf{x} = -2, 6.$$

Therefore, $\textbf{Cont f} = \mathbb{R}_{\mathbf{x}} \setminus \{-2, 6\}$

We will soon learn that f has a **vertical asymptote at x** = - 2 and x = 6.

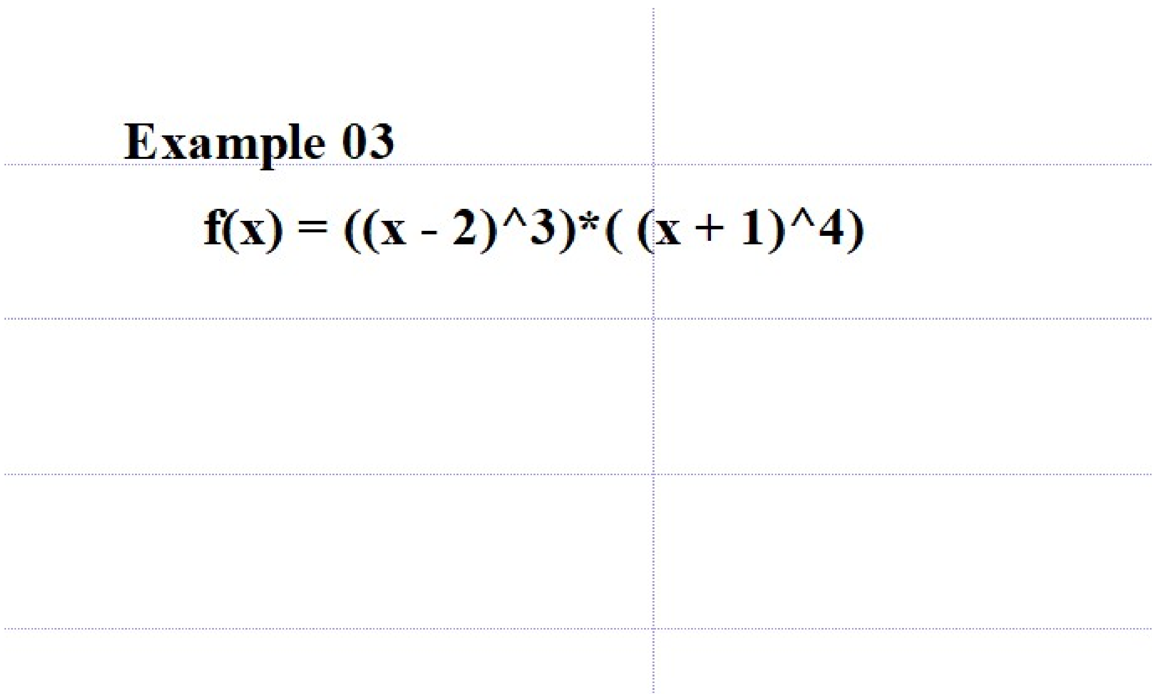


Example 03: Given $\mathbf{f}(\mathbf{x}) = (\mathbf{x}-2)^3(\mathbf{x}+1)^4$, find **Cont f**.

Solution: Since **f** is *just* a polynomial function in factored form

$$\textbf{Cont f} = \mathbb{R}_{\mathbf{x}}$$

Note: There are NO BREAKS in its graph.



Example 04: Given $\mathbf{f}(\mathbf{x}) = \dfrac{1}{\sqrt{7\mathbf{x} - \mathbf{x}^2}}$, find **Cont f** .

Solution: The **square root function** is continuous for **x** values in it domain but we can NEVER divide by zero (0) so we first set note

$$7x\text{-}x^2 = \mathbf{x}(7 - \mathbf{x}) \neq 0 \Rightarrow \mathbf{x} \neq 0, 7$$

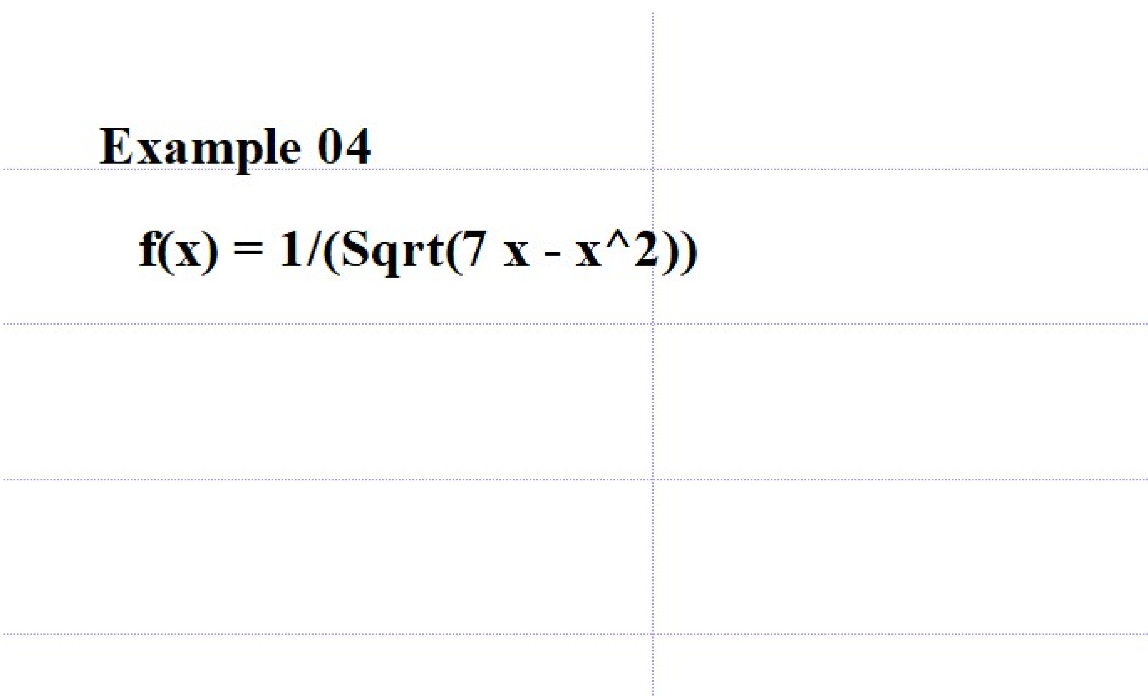
Now $\sqrt{7\mathbf{x} - \mathbf{x}^2}$ is undefined for

$$(-\infty, 0) \cup (7, +\infty)$$

and hence

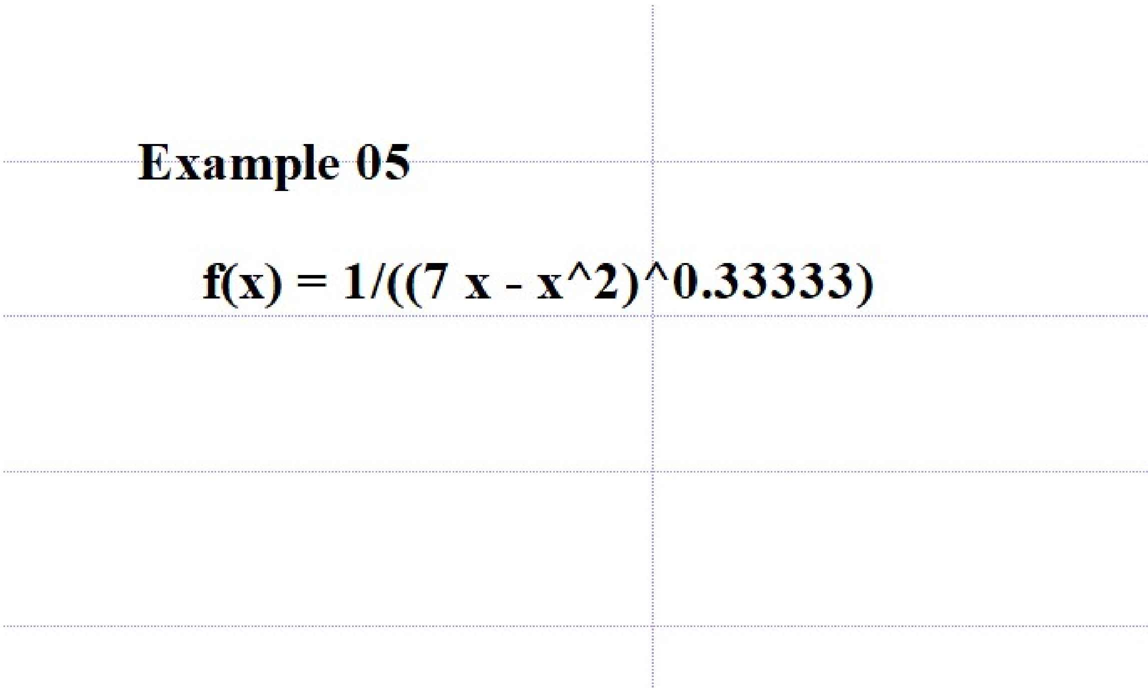
$$\textbf{Cont f} = (0, 7)$$

The graph of **f** is shown below:



Example 05: Given $\mathbf{f}(\mathbf{x}) = \dfrac{1}{\sqrt[3]{7\mathbf{x} - \mathbf{x}^2}}$, find $\mathbf{Cont\ f}$.

Solution: The **cube root function** is continuous for *all* real numbers but again, we can NEVER divide by zero (0) so $\mathbf{x} \neq 0, 7$. Hence $\mathbf{Cont\ f} = \mathbb{R}_{\mathbf{x}} \setminus \{0, 7\}$ and its graph is shown below:



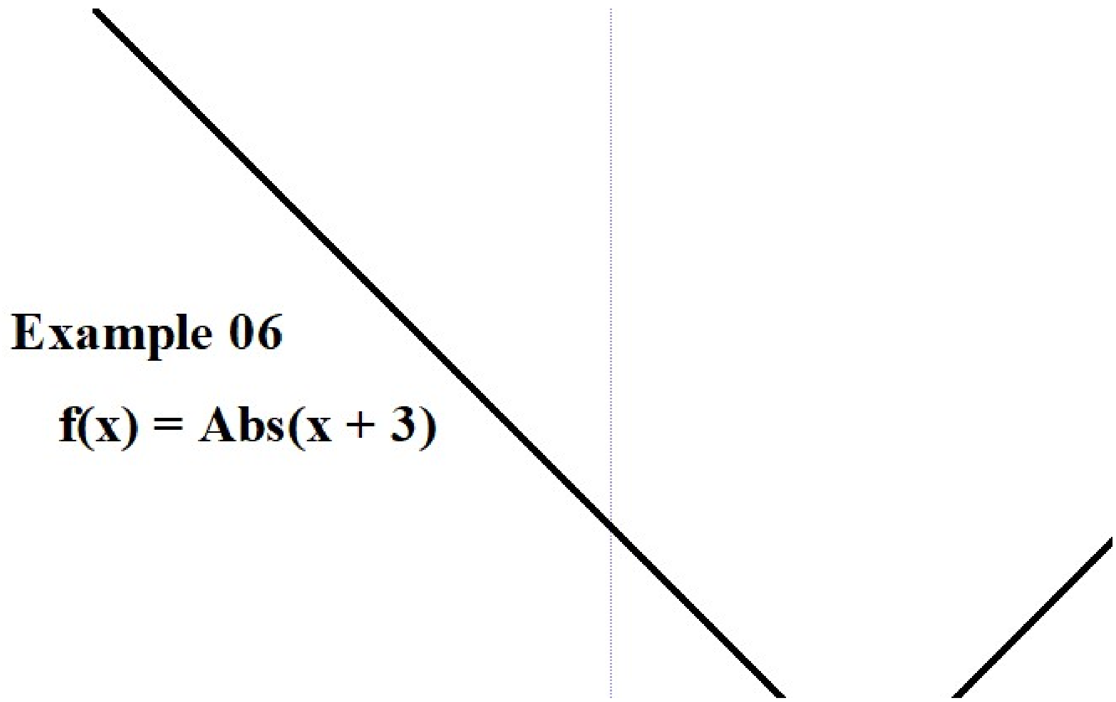
Note: We will see shortly that **f** has **vertical asymptotes** at **x** = 0 and **x**= 7.

Example 06: Given $\mathbf{f}(\mathbf{x}) = |\mathbf{x}+3|$, find **Cont f** .

Solution: The **absolute value function** is continuous at *all* real numbers and **f** is *just* the absolute value function shifted three (3) units to the left:

$$\textbf{Cont f} = \mathbb{R}_{\mathbf{x}}$$

as the graph below illustrates:



Example 07: Given $\mathbf{f}(\mathbf{x})=\begin{cases}\sqrt[3]{\mathbf{x}^3-8} & \textbf{if } \mathbf{x}\in(-\infty,2)\\ 3-2\mathbf{x}^2+\mathbf{x}^3 & \textbf{if } \mathbf{x}\in[2,3]\\ -4\mathbf{x}+24 & \textbf{if } \mathbf{x}\in(3,+\infty)\end{cases}$, find $\textbf{Cont f}$.

Solution: Indirect applications of the continuity theorems yields

$$(-\infty,2)\cup(2,3)\cup(3,+\infty)\subseteq \textbf{Cont f}$$

We must, however, use the definition of continuity at **x** = 2 and **x** = 3 since **f** is defined differently on each side of these **x** values:

Is f continuous at $\mathbf{x}=2$? (NO theorems to help us!)

1. Is $\mathbf{x}_0=2\in\textbf{Dom f}$? YES!
2. Is $\underset{\mathbf{x}\to 2}{\textbf{Lim}}\ \mathbf{f}(\mathbf{x})\in\mathbb{R}$? NO!

$$\mathbf{L}^-=\mathbf{L}_{\mathbf{left}}=\underset{\mathbf{x}\to 2^-}{\textbf{Lim}}\ \mathbf{f}(\mathbf{x})=\underset{\mathbf{x}\to 2^-}{\textbf{Lim}}\ \sqrt[3]{\mathbf{x}^3-8}=0$$

$$\mathbf{L}^+=\mathbf{L}_{\mathbf{right}}=\underset{\mathbf{x}\to 2^+}{\textbf{Lim}}\ \mathbf{f}(\mathbf{x})=\underset{\mathbf{x}\to 2^+}{\textbf{Lim}}\ 3-2\mathbf{x}^2+\mathbf{x}^3=3$$

$$\therefore \underset{\mathbf{x}\to 2}{\textbf{Lim}}\ \mathbf{f}(\mathbf{x})=\textbf{Undefined},\ldots$$

$\therefore$ f is NOT continuous at $\mathbf{x}_0=2$

Is f continuous at $\mathbf{x}=3$? (NO theorems to help us!)

1. Is $\mathbf{x}_0=3\in\textbf{Dom f}$? YES!
2. Is $\underset{\mathbf{x}\to 3}{\textbf{Lim}}\ \mathbf{f}(\mathbf{x})\in\mathbb{R}$? YES!

$$\mathbf{L}^-=\mathbf{L}_{\mathbf{left}}=\underset{\mathbf{x}\to 3^-}{\textbf{Lim}}\ \mathbf{f}(\mathbf{x})=\underset{\mathbf{x}\to 3^-}{\textbf{Lim}}\ 3-2\mathbf{x}^2+\mathbf{x}^3=12$$

$$\mathbf{L}^+=\mathbf{L}_{\mathbf{right}}=\underset{\mathbf{x}\to 3^+}{\textbf{Lim}}\ \mathbf{f}(\mathbf{x})=\underset{\mathbf{x}\to 3^+}{\textbf{Lim}}\ -4\mathbf{x}+24=12$$

$$\therefore \underset{\mathbf{x}\to 3}{\textbf{Lim}}\ \mathbf{f}(\mathbf{x})=12\in\mathbb{R}$$

3. Is $\underset{\mathbf{x}\to 3}{\textbf{Lim}}\ \mathbf{f}(\mathbf{x})=\mathbf{f}(3)$? YES!

$$\mathbf{f}(3)=3-2(3)^2+(3)^3=12$$

$\therefore$ f is continuous at $\mathbf{x}_0=3$

Therefore, **Cont f** $= \mathbb{R}_{\mathbf{x}} \setminus \{2\}$. Note there is a finite jump of "3" units at **x** = 2.

Example 07

$$f(x) = (x^3 - 8)^{(1/3)} \quad \text{if } x \in (-\infty, 2)$$
$$= 3 - 2x^2 + x^3 \quad \text{if } x \in [2, 3]$$
$$= -4x + 24 \quad \text{if } x \in (3, +\infty)$$

This theorem will be applicable through Calculus I:

Theorem (Fundamental Theorem of Continuity): Let $\mathbf{h}$ $(\mathbf{f}\ ;\ \mathbf{f}'\ ;\ \mathbf{f}'')$ be a function defined on an interval $\mathbf{I}$. Assume

1. $\mathbf{h}$ is continuous for all $\mathbf{x} \in \mathbf{I}$ (NO Breaks)
2. $\mathbf{h}(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in \mathbf{I}$ (NO x-intercept POINTS)

Then either

1. $\mathbf{h}(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathbf{I}$
2. $\mathbf{h}(\mathbf{x}) < 0$ for all $\mathbf{x} \in \mathbf{I}$

For now, we will use this theorem to determine where the graph of a function **f** is above (**POS f**) and below (**NEG** f)the **x**-axis.

Example 08: Given $\mathbf{f}(\mathbf{x})=\dfrac{\mathbf{x}^3+2\mathbf{x}^2-5\mathbf{x}-6}{\mathbf{x}^4-\mathbf{x}^3-12\mathbf{x}^2}$, find **Pos f** and **Neg f**.

Solution: The **denominator** easily factors:

$$\mathbf{x}^4-\mathbf{x}^3-12\mathbf{x}^2=\mathbf{x}^2\left(\mathbf{x}^2-\mathbf{x}-12\right)=\mathbf{x}^2\left(\mathbf{x}-4\right)\left(\mathbf{x}+3\right)$$

Factoring this **numerator** requires some skills for Precalculus:

$$\text{Possible Rational Roots: } \mathbf{r}=\frac{\text{Factors of "6"}}{\text{Factors of "1"}}=\frac{\pm6,\pm3,\pm2,\pm1}{\pm1}$$

$$\mathbf{f}(-3)=0\Rightarrow \mathbf{x}+3 \text{ is a factor of } \mathbf{x}^3+2\mathbf{x}^2-5\mathbf{x}-6:$$

$$\begin{aligned}\mathbf{x}^3+2\mathbf{x}^2-5\mathbf{x}-6&=\left(\mathbf{x}+3\right)\left(\mathbf{x}^2-\mathbf{x}-2\right)\\&=\left(\mathbf{x}+3\right)\left(\mathbf{x}-2\right)\left(\mathbf{x}+1\right)\end{aligned}$$

Thus, $\textbf{Dom f}=\mathbb{R}\setminus\left\{-3,0,4\right\}=\textbf{Cont f}$ which implies

$$\begin{aligned}\mathbf{f}(\mathbf{x})&=\frac{\mathbf{x}^3+2\mathbf{x}^2-5\mathbf{x}-6}{\mathbf{x}^4-\mathbf{x}^3-12\mathbf{x}^2}\\&=\frac{\left(\mathbf{x}+3\right)\left(\mathbf{x}-2\right)\left(\mathbf{x}+1\right)}{\mathbf{x}^2\left(\mathbf{x}-4\right)\left(\mathbf{x}+3\right)}\\&=\frac{\left(\mathbf{x}-2\right)\left(\mathbf{x}+1\right)}{\mathbf{x}^2\left(\mathbf{x}-4\right)}\end{aligned}$$

The **Fundamental Theorem of Continuity** implies that **f** has the same sign in each of the intervals defined by

———————— -3 ———————— -1 —— 0 ———————— 2 ———————— 4 ————————

Selecting an **x** value in each interval to determine the sign of **f** in the entire interval yields

$$\textbf{Neg f}=(-\infty,-3)\cup(-3,-1)\cup(2,4)$$
$$\textbf{Pos f}=(-1,2)\cup(4,+\infty)$$

The graphs below show the positive and negative nature of the graph.

Example 07

f(x) = (x^3 + 2 x^2 - 5 x - 6)/(x^4 - x^3 - 12 x^2

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