Infinite Two-sided (One-sided) Limits

$$\lim_{x \to x_0} f(x) = L \ ; \ x_0 \in \mathbb{R} \ ; \ L = \pm\infty \quad \textbf{(Diverge to } \pm\infty\textbf{)}$$

MATH by Wilson
Your Personal Mathematics Trainer
MathByWilson.com

When **f** is a fraction and $x \to x_0 \in \mathbb{R}$, it may happen that the **numerator** of **f** approaches a non-zero $(\neq 0)$ and the **denominator** approaches zero (0)**:**

$$\frac{\to \neq 0}{\to 0 \text{ (either "+" or "-" only)}}$$

This implies that the function is increasing (or decreasing) without bound. Check out the Excel table: $f(x) = \dfrac{1}{x-2}$; $x \to 2$

x	y = 1 / (x - 2)	
1	-1.00	
1.5	-2.00	
1.999	-1,000.00	
1.999999	-1,000,000.00	
1.999999999	-999,999,917.26	Approaching " - Infinity"

2.000000001	999,999,917.26	Approaching " + Infinity"
2.000001	1,000,000.00	
2.001	1,000.00	
2.5	2.00	
3	1.00	

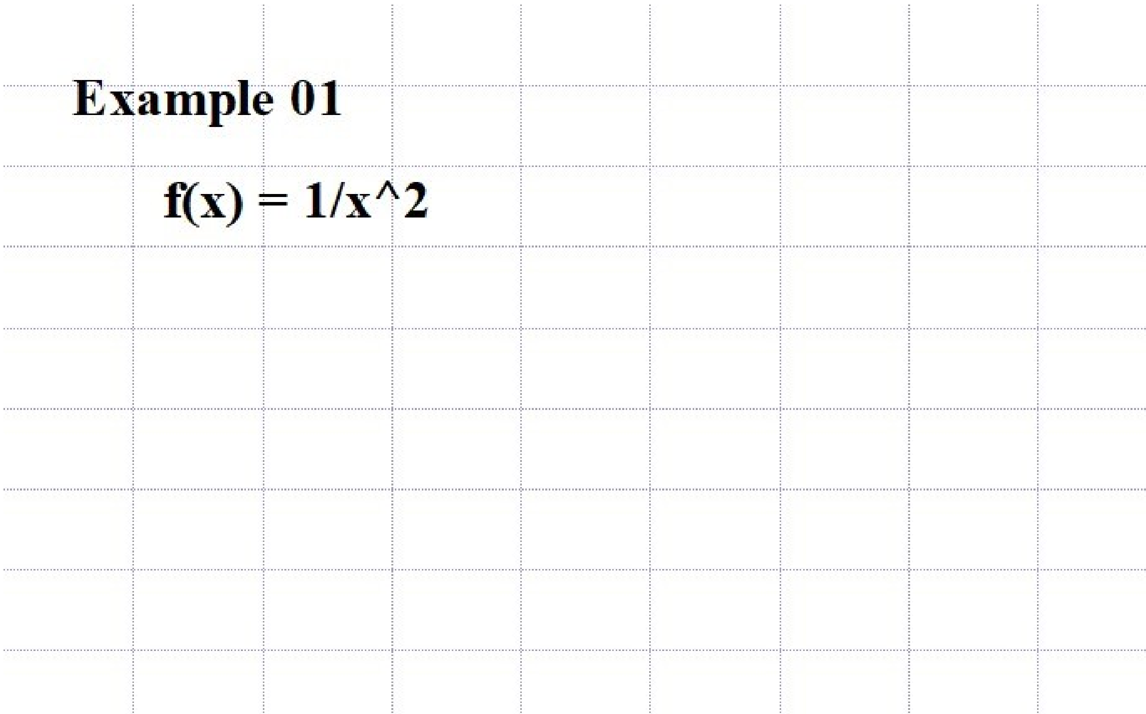
In **"Limit Land"** , we frequently use

$$\frac{1}{\textbf{BIG}} = \textbf{SMALL}$$

$$\frac{1}{\textbf{SMALL}} = \textbf{BIG}$$

Example 01: $\mathbf{\lim_{x \to 0}} \frac{1}{\mathbf{x}^2} = +\infty$ $\left(\textbf{Diverges to } +\infty\right)$ since we have

$$\frac{1}{\mathbf{x}^2} \Rightarrow \frac{1}{\to 0(+)}$$



The vertical line **x** = 0 is called a **vertical asymptote** of **f**.

Example 02: Why is $\underset{x \to 0}{\textbf{Lim}} \frac{1}{\mathbf{x}} = \nexists$ (**Diverges**) ?

Solution: We have

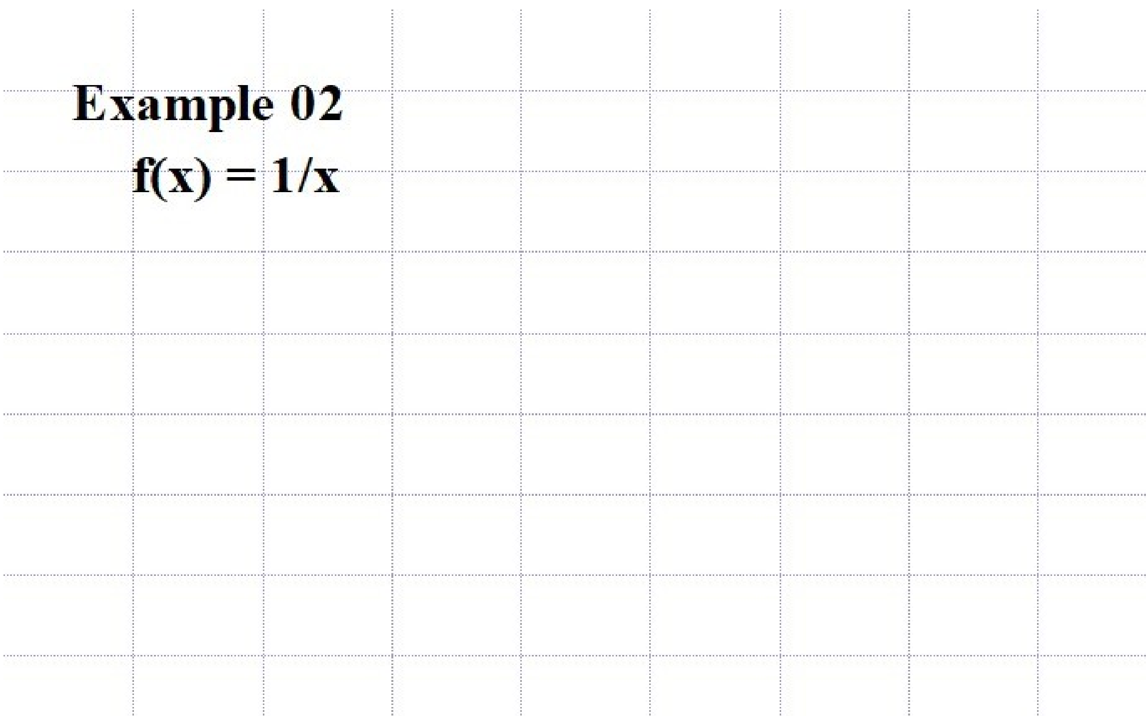
$$\mathbf{L}^{-} = \mathbf{L}_{\text{left}} = \underset{x \to 0^{-}}{\textbf{Lim}} \frac{1}{\mathbf{x}} = -\infty \quad (\textbf{Diverges to } -\infty)$$

$$\text{since } \frac{1}{\mathbf{x}} \Rightarrow \frac{1}{0(-)}$$

$$\mathbf{L}^{+} = \mathbf{L}_{\text{right}} = \underset{x \to 0^{+}}{\textbf{Lim}} \frac{1}{\mathbf{x}} = +\infty \quad (\textbf{Diverges to } +\infty)$$

$$\text{since } \frac{1}{\mathbf{x}} \Rightarrow \frac{1}{0(+)}$$

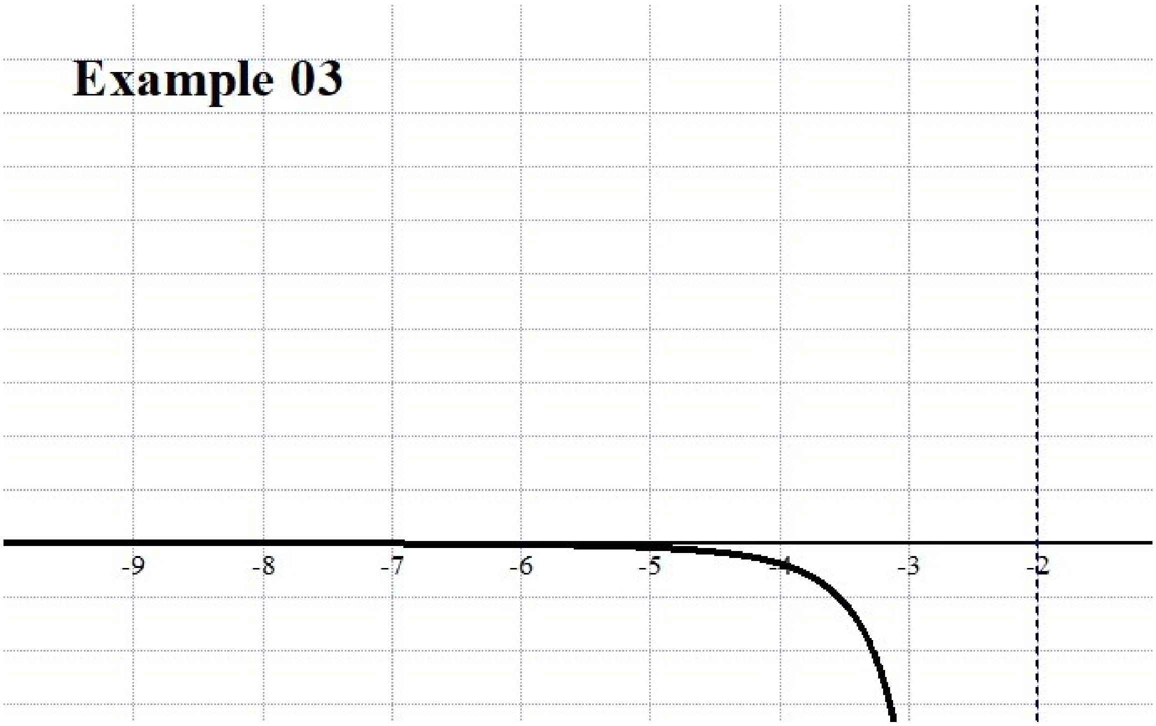
$$\mathbf{L}^{-} \neq \mathbf{L}^{+} \Rightarrow \underset{x \to 0}{\textbf{Lim}} \frac{1}{\mathbf{x}} = \nexists \quad (\textbf{Diverges})$$



The vertical line **x** = 0 is called a **vertical asymptote** of **f**.

Example 03: $\lim\limits_{x\to -2} \dfrac{x-2}{(x+2)^4} = -\infty$ **(Diverges to $-\infty$)** since we have

$$\frac{x-2}{(x+2)^4} \Rightarrow \frac{\to -4}{\to 0(+)}$$



The vertical line **x** = - 2 is called a **vertical asymptote** of **f**

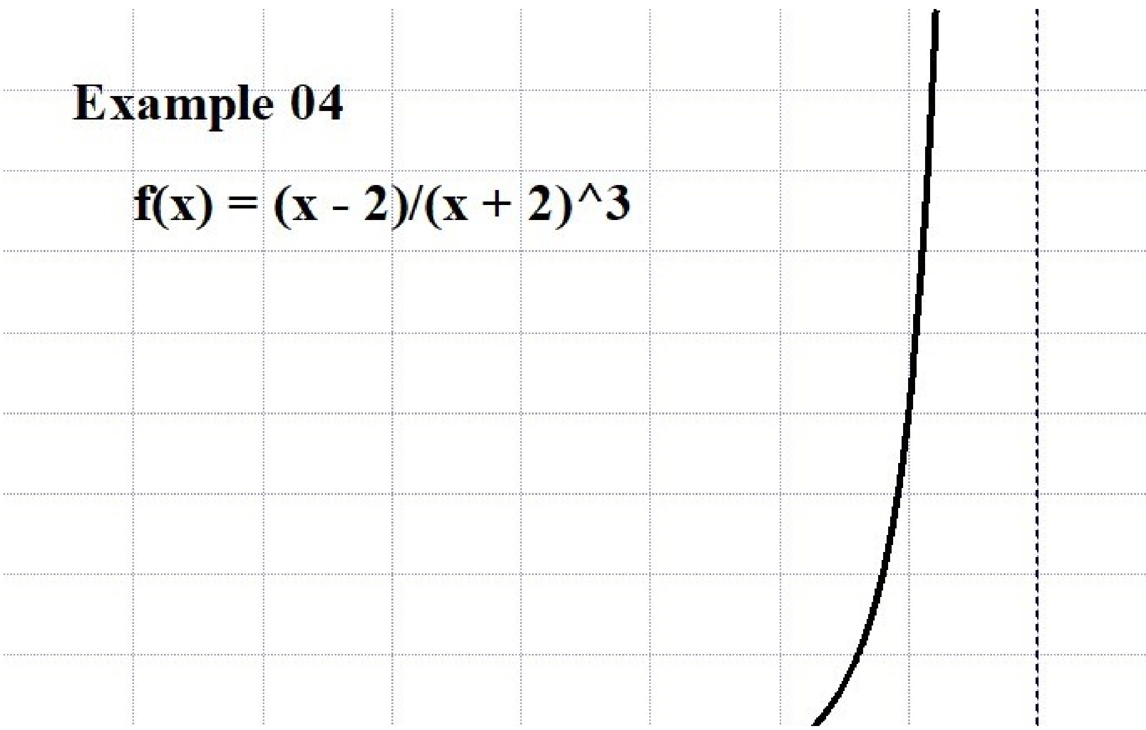
Example 04: Why is $\displaystyle \mathbf{Lim}_{\mathbf{x}\to -2} \frac{\mathbf{x}-2}{(\mathbf{x}+2)^3}$ = Undefined (**Diverges**)?

Since $\displaystyle \mathbf{L}^- = \mathbf{Lim}_{\mathbf{x}\to -2^-} \frac{\mathbf{x}-2}{(\mathbf{x}+2)^3} = +\infty$ (**Diverges to** $+\infty$) $\left\{\frac{\to -4}{\to 0(-)}\right\}$

and

$\displaystyle \mathbf{L}^+ = \mathbf{Lim}_{\mathbf{x}\to -2^+} \frac{\mathbf{x}-2}{(\mathbf{x}+2)^3} = -\infty$ (**Diverges to** $-\infty$) $\left\{\frac{\to -4}{\to 0(+)}\right\}$

we have $\displaystyle \mathbf{Lim}_{\mathbf{x}\to -2} \frac{\mathbf{x}-2}{(\mathbf{x}+2)^3} = \nexists$ (**Diverges**)



= - 2 is called a **vertical asymptote** of **f**

Example 05: Find $\mathbf{\lim\limits_{x\to 0} e^{1/x}}$, if it exists.

Solution:

$$L^- = \lim_{x\to 0^-} e^{1/x} = ?$$

$$x \to 0^- \Rightarrow \frac{1}{x} \to -\infty \Rightarrow e^{1/x} \to 0 \quad (\textbf{Converges to } 0)$$

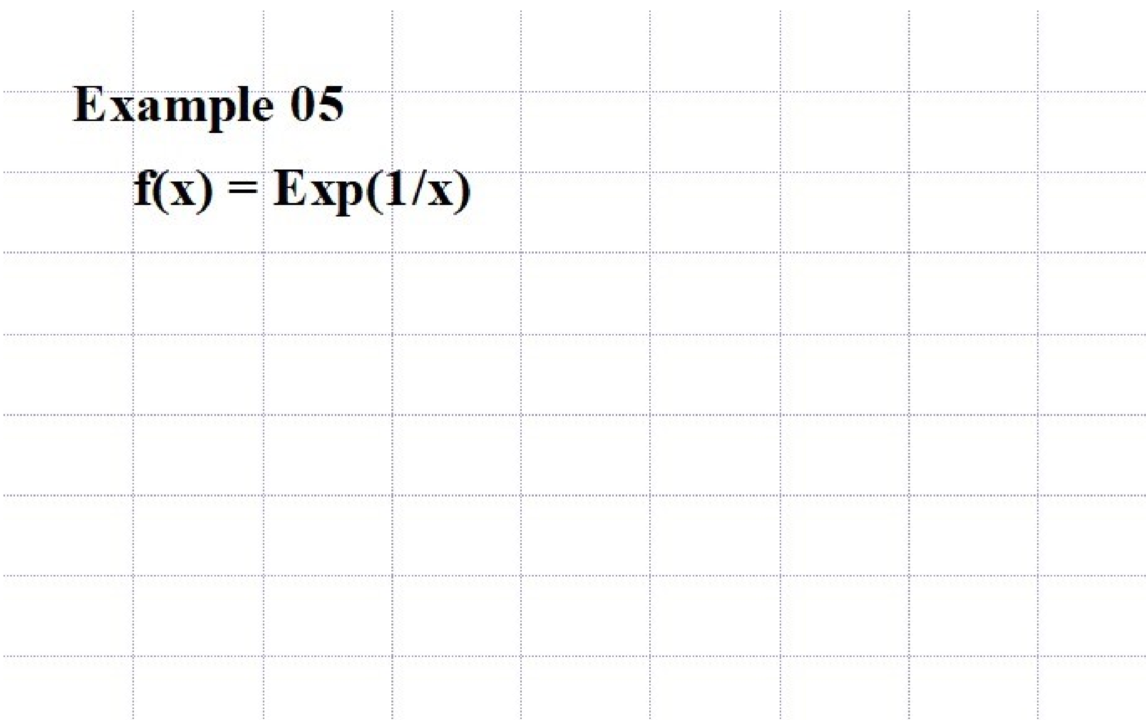
$$\therefore L^- = \lim_{x\to 0^-} e^{1/x} = 0$$

$$L^+ = \lim_{x\to 0^+} e^{1/x} = ?$$

$$x \to 0^+ \Rightarrow \frac{1}{x} \to +\infty \Rightarrow e^{1/x} \to +\infty \quad (\textbf{Diverges to } +\infty)$$

$$\therefore L^+ = \lim_{x\to 0^-} e^{1/x} = +\infty$$

$$\text{Thus } \lim_{x\to 0} e^{1/x} = \nexists \quad (\textbf{Diverges})$$



The vertical line **x** = 0 is called a **vertical asymptote** of **f**

Example 06: Why does $\lim\limits_{x \to 0} \log_{10} \mathbf{x}$ diverge?

Solution: We have

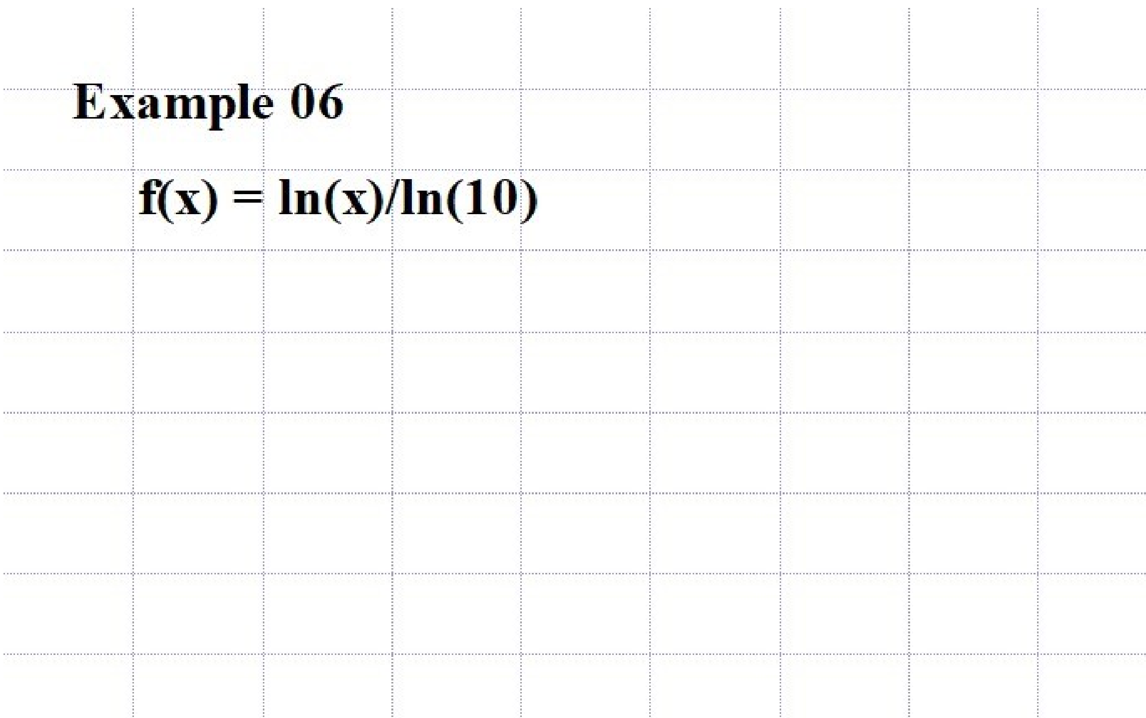
$$\mathbf{L}^{-} = \lim_{x \to 0^{-}} \log_{10} \mathbf{x} = \nexists \quad (\textbf{Diverges})$$

and

$$\mathbf{L}^{+} = \lim_{x \to 0^{+}} \log_{10} \mathbf{x} = -\infty \quad (\textbf{Diverges to } -\infty)$$

since we know the graph of $\log_{10}\mathbf{x}$.

$$\therefore \lim_{x \to 0} \log_{10} \mathbf{x} = \nexists \quad \text{Diverges}$$



The vertical line **x** = 0 is called a **vertical asymptote** of **f**

Example 07: Why is $\lim\limits_{x \to \frac{\pi}{2}} \tan x$ undefined?

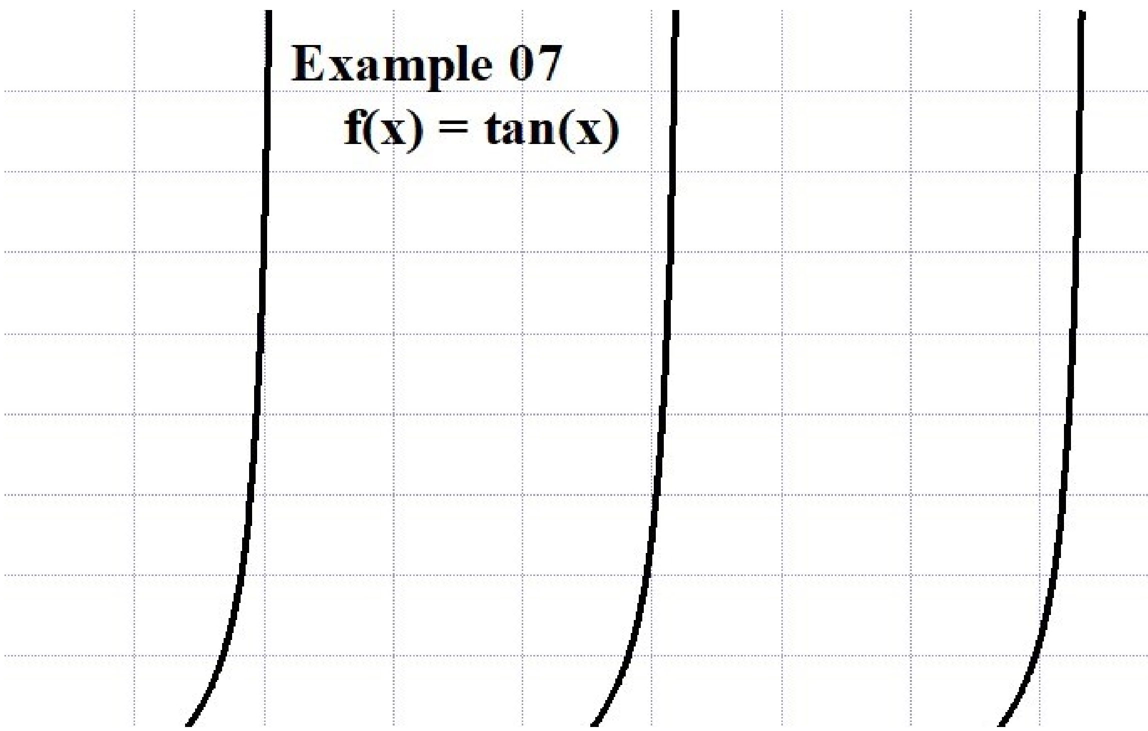
Solution: We know from the graph of the **tangent function**,

$$L^{-} = \lim_{x \to \frac{\pi}{2}^{-}} \tan x = +\infty \qquad (\textbf{Diverges to } +\infty)$$

and

$$L^{+} = \lim_{x \to \frac{\pi}{2}^{+}} \tan x = -\infty \qquad (\textbf{Diverges to } -\infty)$$

$$\therefore \lim_{x \to \frac{\pi}{2}} \tan x = \nexists \qquad (\textbf{Diverges})$$



The line $x = \frac{\pi}{2}$ is *just* one of an infinite number of **vertical asymptotes** the **tangent function** has.

We now present the formal definition of a **vertical asymptote**:

Definition: Let $\mathbf{f}$ be a function. The vertical line $\mathbf{x} = \mathbf{x}_0$ is called a **vertical asymptote** of $\mathbf{f}$ if *at least one* of the following is true:

1. $\lim\limits_{\mathbf{x}\to\mathbf{x}_0^-} \mathbf{f}(\mathbf{x}) = -\infty$
2. $\lim\limits_{\mathbf{x}\to\mathbf{x}_0^-} \mathbf{f}(\mathbf{x}) = +\infty$
3. $\lim\limits_{\mathbf{x}\to\mathbf{x}_0^+} \mathbf{f}(\mathbf{x}) = -\infty$
4. $\lim\limits_{\mathbf{x}\to\mathbf{x}_0^+} \mathbf{f}(\mathbf{x}) = +\infty$

In the next example we will find all the **discontinuities** for a particular function **f**.

Example 08: Given $\mathbf{f}(\mathbf{x}) = \dfrac{\mathbf{x}^3 - 4\mathbf{x}^2 - 12\mathbf{x}}{\mathbf{x}^2 - 10\mathbf{x} + 24}$, find all holes, finite jumps, and vertical asymptotes.

Solution: First of all, recall that rational functions do NOT have finite jumps. Now for the holes and vertical asymptotes, if any. We have

$$\mathbf{Dom\ f} = \mathbb{R}_{\mathbf{x}} \setminus \{4,6\} \text{ since } 0 = \mathbf{x}^2 - 10\mathbf{x} + 24 = (\mathbf{x}-4)(\mathbf{x}-6)$$

and also

$$\mathbf{Cont\ f} = \mathbf{Dom\ f} = \mathbb{R}_{\mathbf{x}} \setminus \{4,6\}$$

There are two (2) breaks in the graph:

$\mathbf{x} = 4$:

$$\lim_{\mathbf{x}\to4^-} \mathbf{f}(\mathbf{x}) = \lim_{\mathbf{x}\to4^-} \frac{\mathbf{x}^3 - 4\mathbf{x}^2 - 12\mathbf{x}}{\mathbf{x}^2 - 10\mathbf{x} + 24} = \lim_{\mathbf{x}\to4^-} \frac{\mathbf{x}(\mathbf{x}-6)(\mathbf{x}+2)}{(\mathbf{x}-4)(\mathbf{x}-6)}$$

$$= \lim_{\mathbf{x}\to4^-} \frac{\mathbf{x}(\mathbf{x}+2)}{(\mathbf{x}-4)} \quad (\mathbf{x} \neq 4,6)$$

$$= -\infty$$

$\therefore \mathbf{x} = 4$ is a vertical asymptote!

Similarily $\lim\limits_{\mathbf{x}\to4^+} \mathbf{f}(\mathbf{x}) = +\infty$

$\mathbf{x} = 6:$

$$\lim_{\mathbf{x}\to 6^-} \mathbf{f}(\mathbf{x}) = \lim_{\mathbf{x}\to 6^-} \frac{\mathbf{x}^3 - 4\mathbf{x}^2 - 12\mathbf{x}}{\mathbf{x}^2 - 10\mathbf{x} + 24} = \lim_{\mathbf{x}\to 6^-} \frac{\mathbf{x}(\mathbf{x}-6)(\mathbf{x}+2)}{(\mathbf{x}-4)(\mathbf{x}-6)}$$

$$= \lim_{\mathbf{x}\to 6^-} \frac{\mathbf{x}(\mathbf{x}+2)}{(\mathbf{x}-4)} \quad (\mathbf{x} \neq 4, 6)$$

$$= 24$$

Similarily $\lim_{\mathbf{x}\to 6^+} \mathbf{f}(\mathbf{x}) = 24$

$\therefore$ the point $(6, 24)$ is a hole!

The graph shows both the vertical asymptote line and a "x" for the hole: