FUNdamental Limit Question

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Drawing and understanding a graph is much more than connecting a few points with straight lines – like in a dot-to-dot coloring book! Technology can be also used to obtain a portion of a graph but do we really understand what we see? Doubtfully! Many years ago I "tore up" my right knee playing racket sports and went to see Dr. P for an evaluation and an x-ray. He showed me the x-ray and asked if I saw the problem. We were looking at the same "graph" but he understood, I did not. I chose him to repair my knee since I did not really understand the problem, much less have the credentials to fix it. We can gain a partial understanding of graphs by studying limits of functions.

FUNdamental Limit Question: If the x values have a pattern as we approach $\mathbf{x_0}$

$$\mathbf{x} \overset{\text{Approach}}{\rightarrow} \mathbf{x_0} = \begin{cases} \text{Number (One or Two-Sided)} \\ \pm\,\infty \end{cases}$$

do the corresponding $\mathbf{f(x)}$ values have a pattern

$$\mathbf{f(x)} \overset{\text{Approach}}{\rightarrow} \mathbf{L} = ????? \ (\mathbf{L} \text{ can be a number or } \pm\,\infty)$$

It is critical to understand that to have a limit, **ALL** the f(x) values MUST BE APROACHING THE SAME THING!

FUNdamental Limit Question (in symbols): $\mathbf{x} \rightarrow \mathbf{x_0} \overset{\text{Implies}}{\Rightarrow} \mathbf{f(x)} \rightarrow \mathbf{L}\ ?????$

In Calculus Notation: $\underset{\mathbf{x}\rightarrow\mathbf{x_0}}{\mathbf{Lim}}\ \mathbf{f(x)} = \mathbf{L}\ ?????$

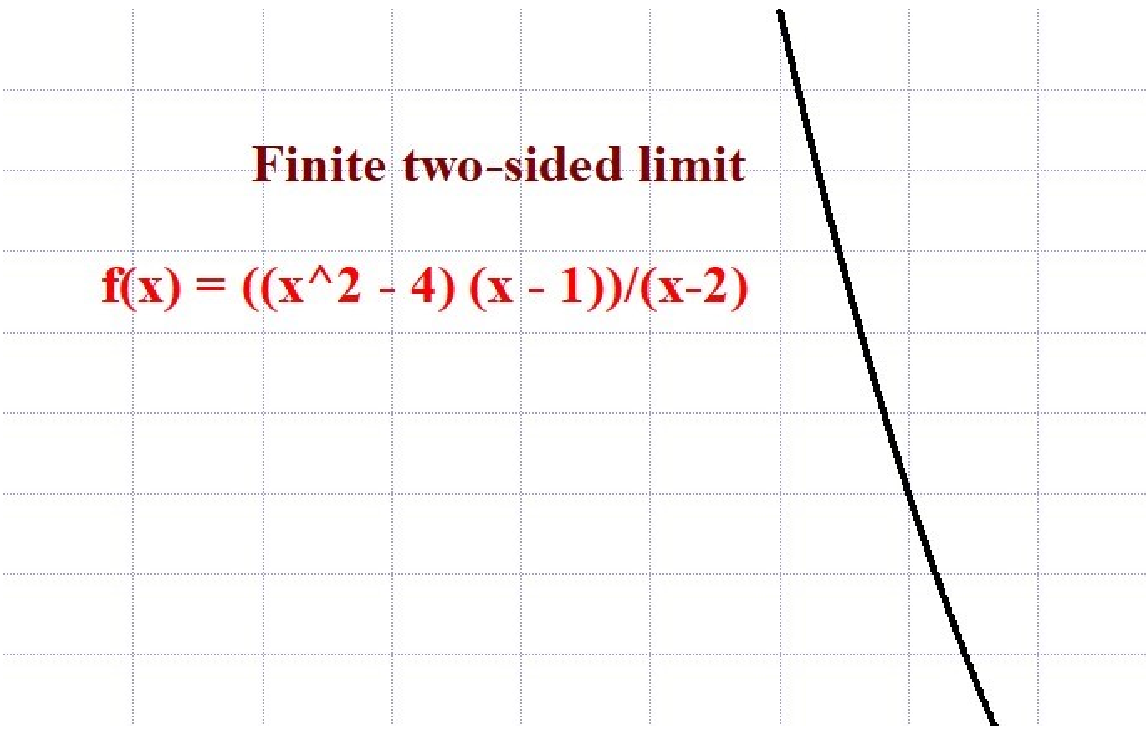
There are four (4) types of options we consider:

1. Finite Two-sided (& One-sided) Limits: $\mathbf{x_0} \in \mathbb{R}_\mathbf{x}$; $\mathbf{L} \in \mathbb{R}_\mathbf{y}$
   a. Two-sided: $\mathbf{x_0} \in \mathbb{R}_\mathbf{x}$
   b. One-sided: $\mathbf{x_0} \in \mathbb{R}_\mathbf{x}$
      i. Left: $\mathbf{x_0}^{-} \in \mathbb{R}_\mathbf{x}$ $[\mathbf{x_0}^{\mathbf{L}}]$
      ii. Right: $\mathbf{x_0}^{+} \in \mathbb{R}_\mathbf{x}$ $[\mathbf{x_0}^{\mathbf{R}}]$

Consider the function rational $\mathbf{f}(\mathbf{x}) = \frac{(\mathbf{x}^2 - 4)(\mathbf{x} - 1)}{(\mathbf{x} - 2)}$

First note that $\mathbf{x} = 2$ is **NOT** in the domain of the function. Also note that as the x values as chosen "closer and closer" to $\mathbf{x} = 2$ (BUT NEVER EQUAL TO "2"), the corresponding f(x) values get "closer and closer" to "4" on the **y**-axis. We call "4" the limit and write

$$\lim_{\mathbf{x} \to 2} \mathbf{f}(\mathbf{x}) = 4$$



The actual goal is **NOT** to look at the graph and "see" what appears to be going on, BUT to develop analytical tools – forthcoming – to first determine what the graph has to be doing and use that information to construct the appropriate portion of the graph.

Consider the function $\mathbf{f}(\mathbf{x}) = \frac{4|\mathbf{x}|}{\mathbf{x}}$, an absolute value related function.

First note that $\mathbf{x} = 0$ is **NOT** in the domain of the function. In this example as the x values are chosen "closer and closer" to $\mathbf{x} = 0$ (BUT NEVER EQUAL TO "0"), the corresponding f(x) values are **NOT** getting "closer and closer" to some unique number so that

$$\lim_{x \to 0} f(x) = \text{ Undefined, does not exist, ...}$$

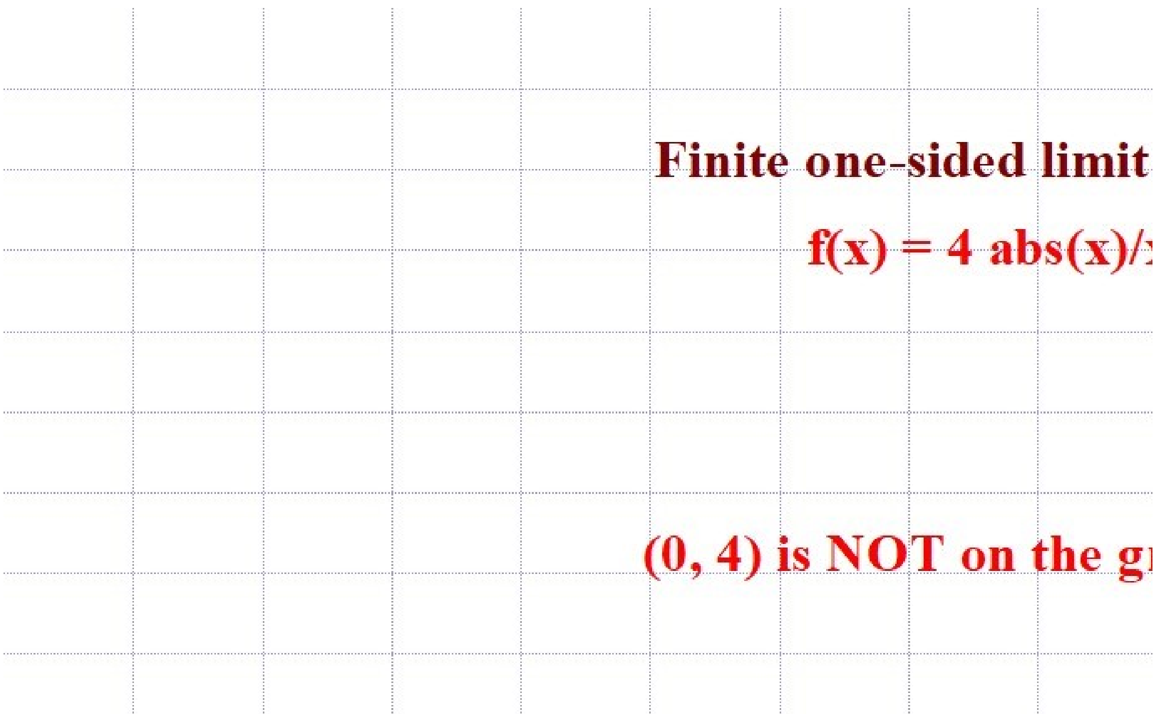
However, if we choose x values “closer and closer” to $x = 0$ but less than 0 (to the **“left”** we say), the corresponding f(x) values are getting “closer and closer” to “- 4” Actually, in this particular case, they are actually equal to “- 4” – now that’s close. Hence we write

$$\lim_{x \to 0^-} f(x) = -4$$

This is called the **limit from the left** and the “-“ in x_0^-, whatever x_0 is, just means that we are choosing numbers less than x_0. It has NOTHING to do with whether we are choosing positive or negative numbers!

In a similar manner, we have

$$\lim_{x \to 0^+} f(x) = +4$$



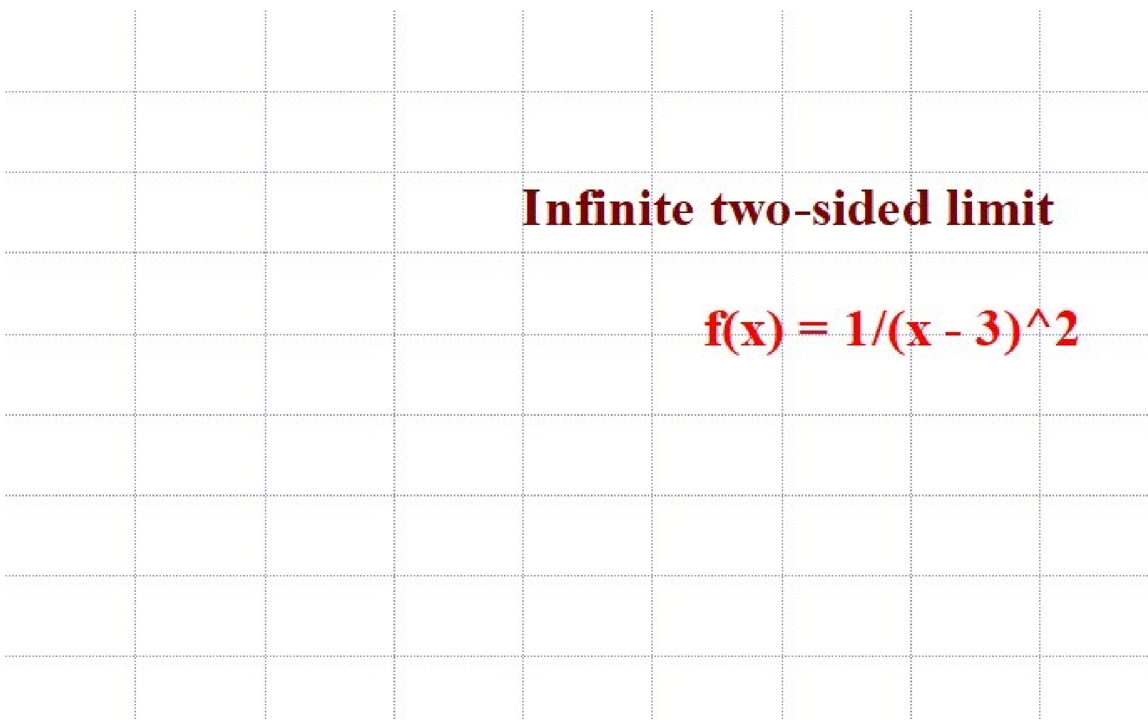
Again, our actual goal is to develop analytical tools to first determine what the graph has to be doing and use that information to construct the appropriate portion of the graph.

2. Infinite Two-sided (& One-sided) Limits: $\mathbf{x}_0 \in \mathbb{R}_{\mathbf{x}}; \mathbf{L} = \pm\infty$
   a. Two-sided: $\mathbf{x}_0 \in \mathbb{R}_{\mathbf{x}}$
   b. One-sided: $\mathbf{x}_0 \in \mathbb{R}_{\mathbf{x}}$
      i. Left: $\mathbf{x}_0^{-} \in \mathbb{R}_{\mathbf{x}}$ $[\mathbf{x}_0^{\mathbf{L}}]$
      ii. Right: $\mathbf{x}_0^{+} \in \mathbb{R}_{\mathbf{x}}$ $[\mathbf{x}_0^{\mathbf{R}}]$

Consider the function $\mathbf{f}(\mathbf{x}) = \dfrac{1}{(\mathbf{x}-3)^2}$, the basic $\dfrac{1}{\mathbf{x}^2}$ function shifted three (3) units to the right. First note that $\mathbf{x} = 3$ is **NOT** in the domain of the function. Also note that as the x values as chosen "closer and closer" to $\mathbf{x} = 3$ (BUT NEVER EQUAL TO "3"), the corresponding f(x) values are increasing without bound through positive values. Hence, we have an infinite two-sided limit and write

$$\lim_{\mathbf{x}\to 3} \mathbf{f}(\mathbf{x}) = +\infty$$

The vertical line $\mathbf{x} = 3$ is called a **vertical asymptote** of the function.



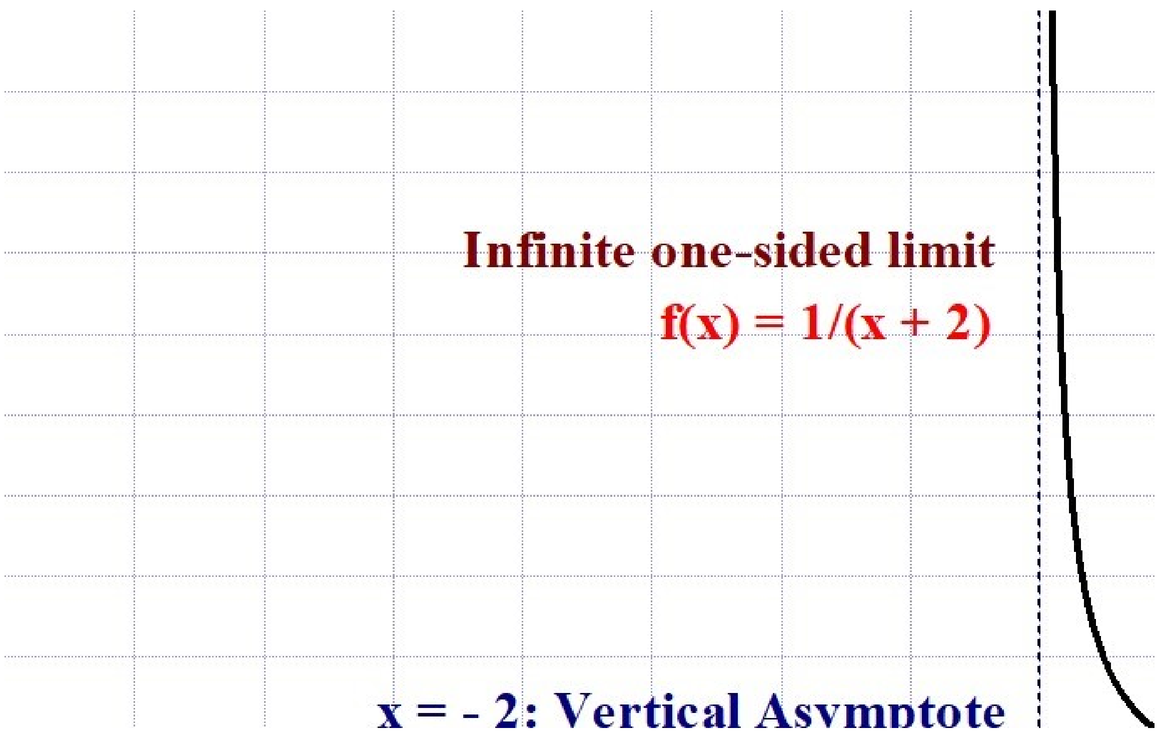
Consider the function $\mathbf{f}(\mathbf{x}) = \dfrac{1}{\mathbf{x}+2}$, the basic $\dfrac{1}{\mathbf{x}}$ function shifted two (2) units to the left. First note that $\mathbf{x} = -2$ is **NOT** in the domain of the function.
As the x values as chosen "closer and closer" to $\mathbf{x} = -2$ from the "left", the corresponding f(x) values are decreasing without bound through negative values but are increasing without bound from the "right. Hence, we have two (2) infinite one-sided limits and write

$$\lim_{\mathbf{x}\to -2^-} \mathbf{f}(\mathbf{x}) = -\infty$$

$$\lim_{\mathbf{x}\to -2^+} \mathbf{f}(\mathbf{x}) = +\infty$$

The vertical line $\mathbf{x} = -2$ is called a **vertical asymptote** of the function. Also,

$$\lim_{\mathbf{x}\to -2} \mathbf{f}(\mathbf{x}) = \text{Undefined, does not exist, ...}$$



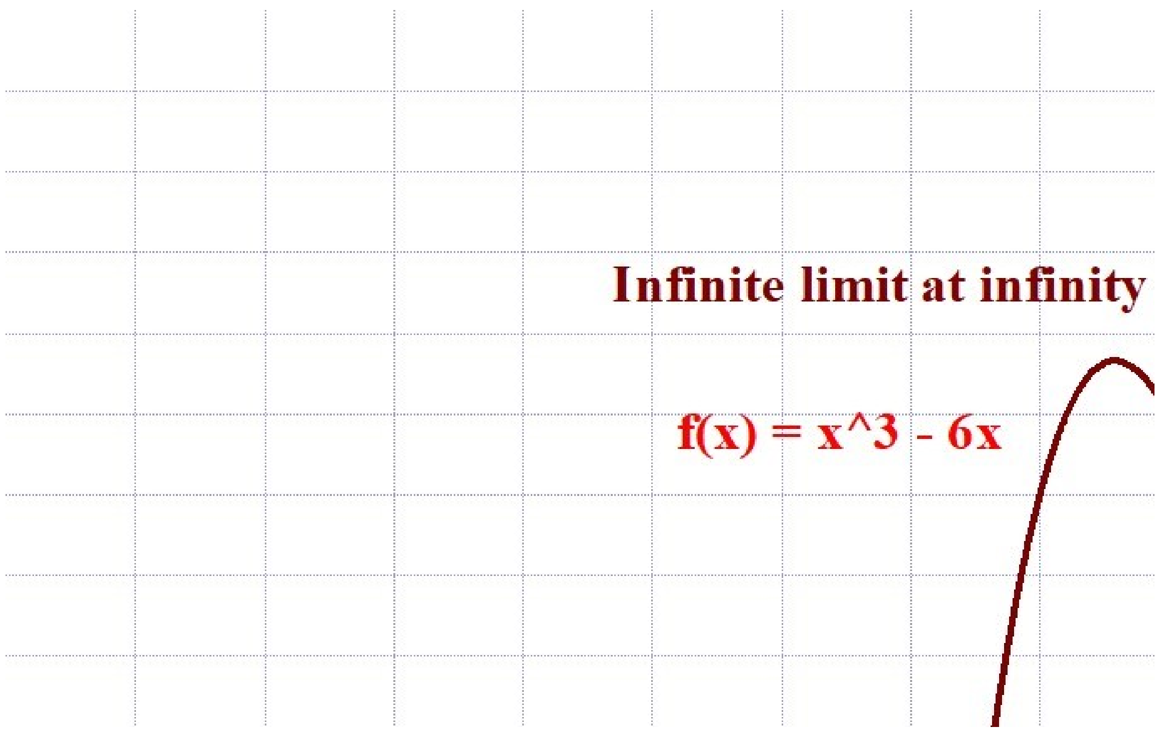
3. Infinite Limits at Infinity: $\mathbf{x}_0 = \pm\infty; \mathbf{L} = \pm\infty$

Consider the polynomial function $\mathbf{f}(\mathbf{x}) = \mathbf{x}^3 - 6\mathbf{x}$. This time we first choose our values for **x** "larger and larger" without bound. The corresponding f(x) values are also increasing without bound through positive values, but this time they are **y**-values. We call this an infinite limit at infinity and write

$$\lim_{\mathbf{x}\to+\infty} \mathbf{f}(\mathbf{x}) = +\infty$$

Now, we choose our values for **x** "smaller and smaller" without bound. The corresponding f(x) values are decreasing without bound through negative values. We also call this an infinite limit at infinity and write

$$\lim_{\mathbf{x}\to-\infty} \mathbf{f}(\mathbf{x}) = -\infty$$



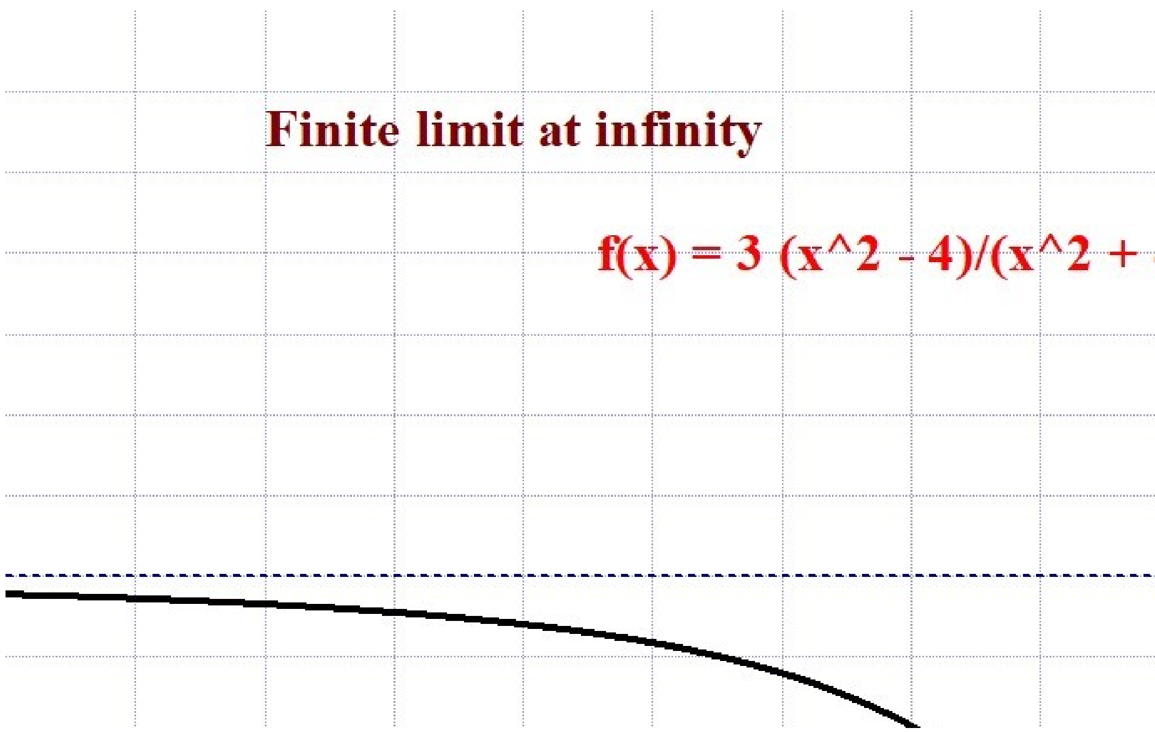
4. Finite Limits at Infinity: $\mathbf{x}_0 = \pm\infty; \mathbf{L} \in \mathbb{R}_{\mathbf{y}}$

Consider the rational function $\mathbf{f}(\mathbf{x}) = \dfrac{3\left(\mathbf{x}^2 - 4\right)}{\mathbf{x}^2 + 4}$. We choose our values for **x** "larger and larger" without bound. The corresponding f(x) values are also approaching "3". We call this a finite limit at infinity and write

$$\lim_{\mathbf{x}\to+\infty} \mathbf{f}(\mathbf{x}) = 6$$

The horizontal line $\mathbf{y} = 6$ is called a **horizontal asymptote** of the function. Also, as we choose our values for **x** "smaller and smaller" without bound. The corresponding f(x) values also approach "6". We also call this a finite limit at infinity and write

$$\lim_{\mathbf{x}\to-\infty} \mathbf{f}(\mathbf{x}) = 6$$



If $\mathbf{L} \in \mathbb{R}_{\mathbf{y}}$, we say the limit **converges**. Otherwise we say the limit **diverges**. If $\mathbf{L} = \pm\infty$, we say the limit **diverges to** $\pm\infty$.

Remember $\pm\infty$ does not represent a number.