Finite Two-sided (& One-sided) Limits
Part I

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For each type of limit, there is a technical definition and from it, many Limit Theorems can be proved to simplify our analysis of functions. This study forms part of an Advanced Calculus class. This being the case, we will concentrate on using the theorems and learning numerous techniques in finding particular limits. Relative to finding finite two-sided limits, finding $\mathbf{\underset{x \to x_0}{Lim}\, f(x) = L}$; $\mathbf{x_0} \in \mathbb{R}$; $\mathbf{L} \in \mathbb{R}$ is trivial if $\mathbf{x_0} \in \mathbf{Dom\ f}$

when

a. **f** is a polynomial
b. **f** is a rational function
c. **f** is … many other types of functions

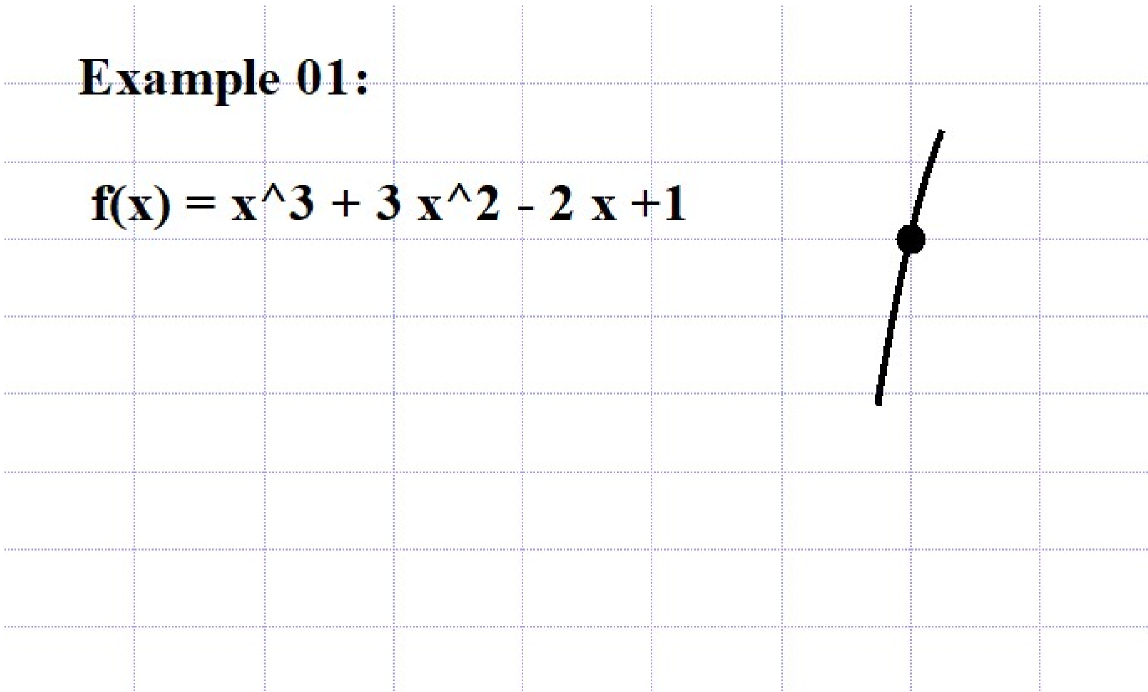
In these cases, the limit $\mathbf{L}$ is just $\mathbf{f(x_0)}$, that is,

$$\mathbf{\underset{x \to x_0}{Lim}} \begin{Bmatrix} \text{polynomial} \\ \text{rational} \\ \ldots \end{Bmatrix} = \mathbf{f(x_0)}$$

One more time: Just evaluate **f** at $\mathbf{x_0}$ and we have the limit: $\mathbf{L = f(x_0)}$

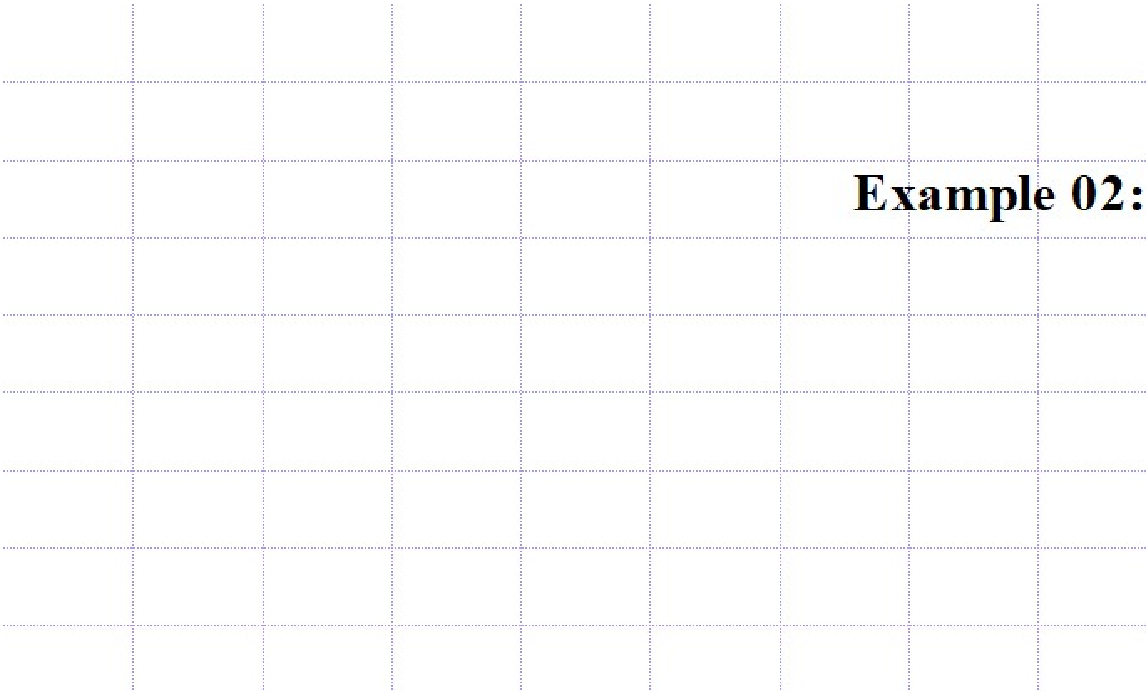
Example 01: Find $\lim\limits_{x \to -3} x^3 + 3x^2 - 2x + 1 \overset{?}{=} L$

Solution: We have $L = f(-3) = 7$; **Converges** : "**C**"



Example 02: Find $\lim\limits_{x \to 2} \dfrac{x^2 - 2x + 2}{x + 2} \overset{?}{=} L$

Solution: We have $L = f(2) = \dfrac{1}{2}$; **Converges** : "**C**"



It is NOT always this easy … sorry about that! Under certain circumstances, the Quotient Limit Theorem states

$$\lim\limits_{x \to x_0} \frac{g(x)}{h(x)} = \frac{\lim\limits_{x \to x_0} g(x)}{\lim\limits_{x \to x_0} h(x)}$$

But when $\lim\limits_{x \to x_0} g(x) = 0$ and $\lim\limits_{x \to x_0} h(x) = 0$, we get $\dfrac{0}{0}$ which is called an **indeterminate form** (**IF**). When this occurs, it means that we have not done the right thing yet! *Factoring* frequently plays a key role in the right thing to do: We change the form of $\dfrac{g(x)}{h(x)}$ to determine what the limit actually is. FYI: We will see other **IF**s as we progress!

Example 03: Find $\mathbf{\lim\limits_{x\to 2}} \dfrac{\mathbf{x}^3 - 4\mathbf{x}^2 + 3\mathbf{x} + 2}{\mathbf{x} - 2} \overset{?}{=} \mathbf{L}$

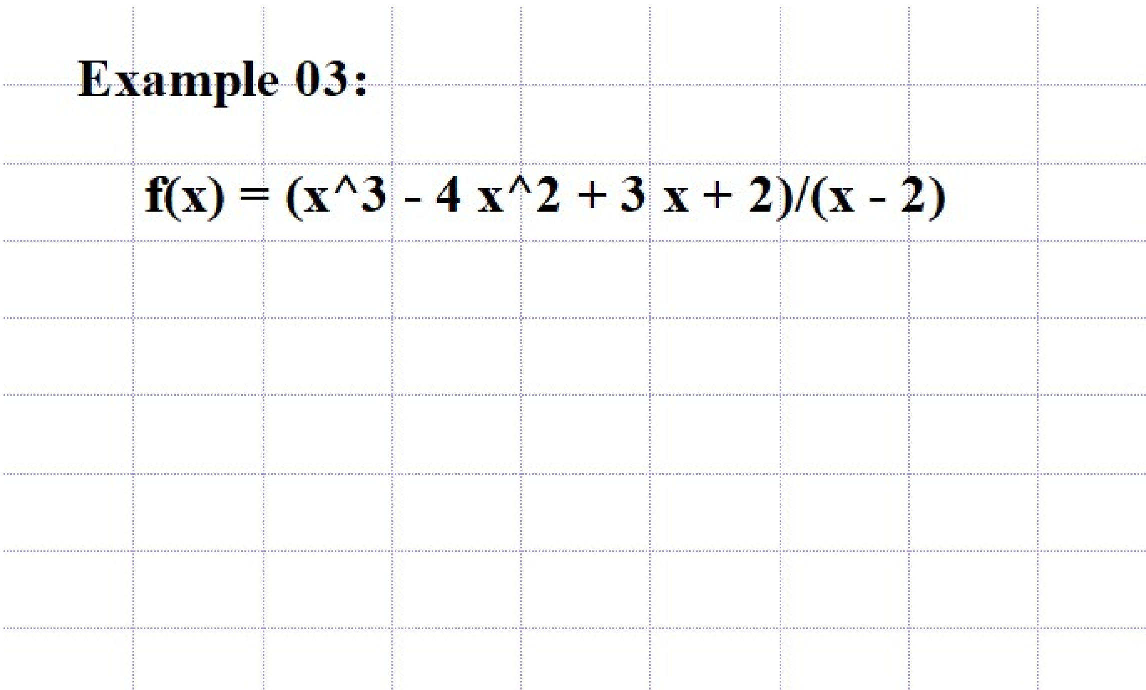
Solution: We have $\mathbf{\lim\limits_{x\to 2}} \dfrac{\mathbf{x}^3 - 4\mathbf{x}^2 + 3\mathbf{x} + 2}{\mathbf{x} - 2} \Rightarrow \dfrac{\approx 0}{\approx 0}$ so we need to change the form.

Hence,

$$\mathbf{\lim_{x\to 2}} \frac{\mathbf{x}^3 - 4\mathbf{x}^2 + 3\mathbf{x} + 2}{\mathbf{x} - 2} = \mathbf{\lim_{x\to 2}} \frac{(\mathbf{x} - 2)(\mathbf{x}^2 - 2\mathbf{x} - 1)}{\mathbf{x} - 2} \quad \{\text{Factor}\}$$

$$\underset{\mathbf{x}\neq 2}{=} \mathbf{\lim_{x\to 2}} \left(\mathbf{x}^2 - 2\mathbf{x} - 1\right)$$

$$= \mathbf{L} = -1 \qquad \text{Converges: "C"}$$



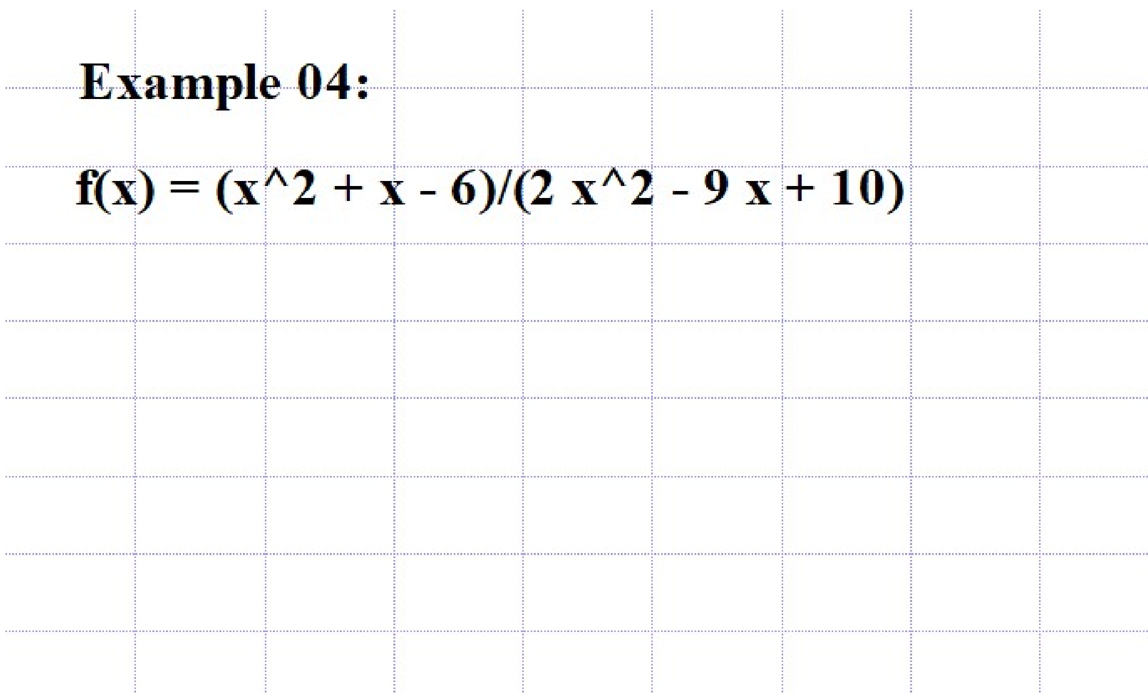
Example 04: Find $\lim\limits_{x \to 2} \dfrac{x^2 + x - 6}{2x^2 - 9x + 10} \stackrel{?}{=} L$

Solution: Since $\lim\limits_{x \to 2} \dfrac{x^2 + x - 6}{2x^2 - 9x + 10} \Rightarrow \dfrac{\approx 0}{\approx 0}$, we must change the form:

$$\lim_{x \to 2} \frac{x^2 + x - 6}{2x^2 - 9x + 10} = \lim_{x \to 2} \frac{(x-2)(x+3)}{(x-2)(2x-5)} \ \{\text{Factor}\}$$

$$\underset{x \neq 2}{=} \lim_{x \to 2} \frac{x+3}{2x-5}$$

$$= L = -5 \qquad \text{Converges: "C"}$$



Example 05: Find $\lim\limits_{x\to 1}\dfrac{\sqrt{-2x^2+3x}-1}{x-1}\overset{?}{=}L$

Solution: We obtain $\dfrac{0}{0}$ as before but we must "rationalize" before we "factor":

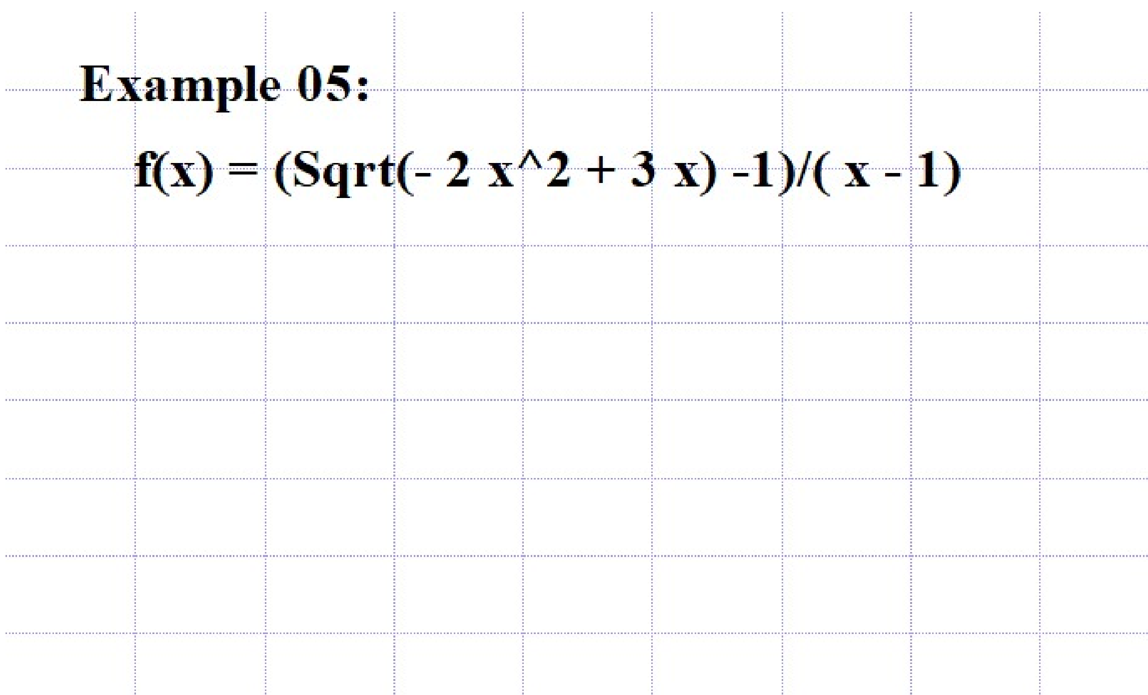
$$\lim_{x\to 1}\frac{\sqrt{-2x^2+3x}-1}{x-1}=\lim_{x\to 1}\frac{\sqrt{-2x^2+3x}-1}{x-1}\;\frac{\sqrt{-2x^2+3x}+1}{\sqrt{-2x^2+3x}+1}$$

$$=\lim_{x\to 1}\;\frac{-2x^2+3x-1}{(x-1)\left(\sqrt{-2x^2+3x}+1\right)}$$

$$=\lim_{x\to 1}\;\frac{-(x-1)(2x-1)}{(x-1)\left(\sqrt{-2x^2+3x}+1\right)}\;\{\text{Factor}\}$$

$$\underset{x\neq 1}{=}\;\lim_{x\to 1}\frac{-(2x-1)}{\sqrt{-2x^2+3x}+1}$$

$$=L=-\frac{1}{2}\qquad \text{Converges: "C"}$$



Example 06: Find $\lim\limits_{x\to 4} \dfrac{\frac{1}{4}-\frac{1}{x}}{x-4} \stackrel{?}{=} L$

Solution: We have $\lim\limits_{x\to 4} \dfrac{\frac{1}{4}-\frac{1}{x}}{x-4} \Rightarrow \dfrac{\approx 0}{\approx 0}$; we must change the form:

$$\lim_{x\to 4} \frac{\frac{1}{4}-\frac{1}{x}}{x-4} = \lim_{x\to 4} \frac{\frac{1}{4}-\frac{1}{x}}{x-4}\,\frac{4x}{4x}$$

$$= \lim_{x\to 4} \frac{x-4}{4x(x-4)} \quad \{\text{Factor}\}$$

$$\underset{x\neq 4}{=} \lim_{x\to 4} \frac{1}{4x}$$

$$= L = \frac{1}{16} \qquad \text{Converges: "C"}$$

Example 06:

f(x) = (1/4 - 1/x) / (x - 4)

Example 07: Find $\mathbf{f}(\mathbf{x}) = \begin{cases} \dfrac{\mathbf{x}^2 - 9}{\mathbf{x} - 3} \; ; \mathbf{x} \in \mathbb{R} \setminus \{3\} \\ \mathbf{f}(3) = 7 \end{cases}$

Solution: We have

$$\underset{\mathbf{x}\to 3}{\mathbf{Lim}}\ \mathbf{f}(\mathbf{x}) \underset{\mathbf{x}\neq 3}{=} \underset{\mathbf{x}\to 3}{\mathbf{Lim}} \frac{\mathbf{x}^2 - 9}{\mathbf{x} - 3} \left\{ \Rightarrow \frac{0}{0} \Rightarrow \text{Change the form} \right\}$$

$$\underset{\mathbf{x}\neq 3}{=} \underset{\mathbf{x}\to 3}{\mathbf{Lim}} \frac{(\mathbf{x} - 3)(\mathbf{x} + 3)}{\mathbf{x} - 3} \left\{ \text{Factor} \right\}$$

$$\underset{\mathbf{x}\neq 3}{=} \underset{\mathbf{x}\to 3}{\mathbf{Lim}} \frac{\mathbf{x} + 3}{1}$$

$$= \mathbf{L} = 6 \qquad \text{Converges: "C"}$$