1Finite Two-sided (and One-sided) Limits
Part III

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Sometimes it is convenient or even necessary to consider one-sided limits.

a. Left: **x** values chosen less than $\mathbf{x}_0$:

$$\lim_{\mathbf{x}\to\mathbf{x}_0^-} \mathbf{f}(\mathbf{x}) = \mathbf{L}_{\mathbf{left}} \ ; \mathbf{x}_0 \in \mathbb{R} \ ; \mathbf{L}_{\mathbf{left}} \in \mathbb{R}$$

b. Right: **x** values chosen greater than $\mathbf{x}_0$:

$$\lim_{\mathbf{x}\to\mathbf{x}_0^+} \mathbf{f}(\mathbf{x}) = \mathbf{L}_{\mathbf{right}} \ ; \mathbf{x}_0 \in \mathbb{R} \ ; \mathbf{L}_{\mathbf{right}} \in \mathbb{R}$$

Good News: When and only when $\mathbf{L}_{\mathbf{left}} = \mathbf{L}_{\mathbf{right}}$

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \mathbf{f}(\mathbf{x}) = \mathbf{L} = \mathbf{L}_{\mathbf{left}} = \mathbf{L}_{\mathbf{right}} \ ; \mathbf{x}_0 \in \mathbb{R} \ ; \mathbf{L} \in \mathbb{R}$$

Example 01: Analyze $\lim\limits_{x\to 0}\sqrt{x}$

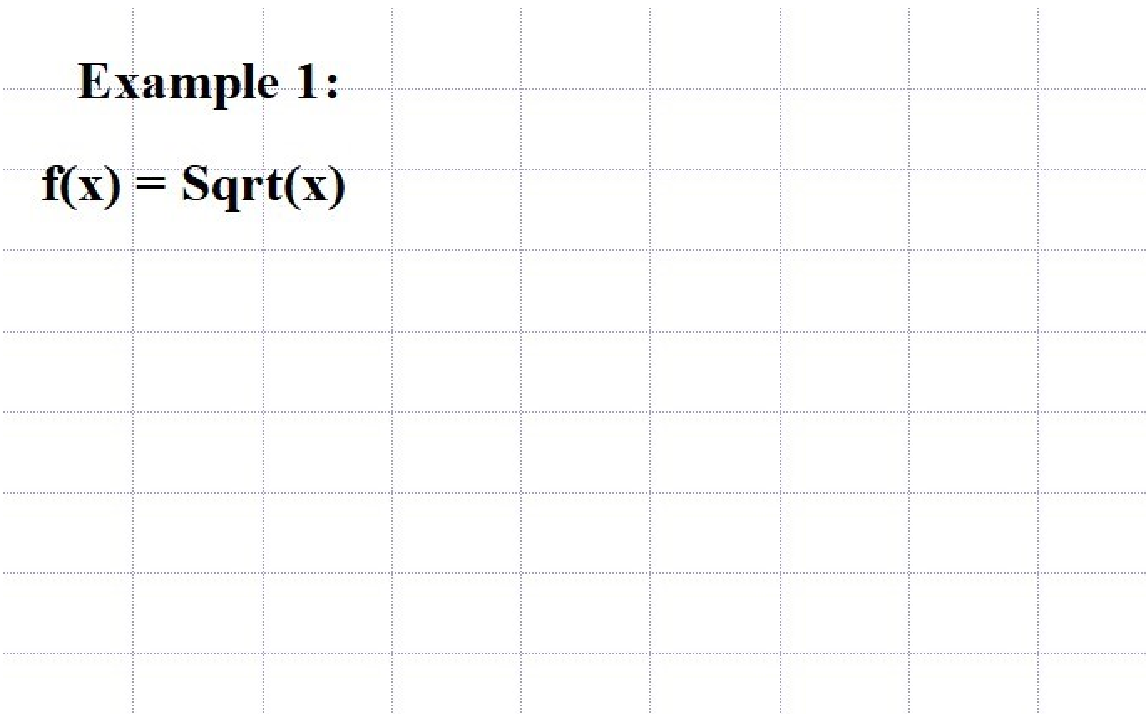
Solution: Since the domain satisfies $x \geq 0$,

$$\lim_{x\to 0^-}\sqrt{x} = \text{undefined, do not exist, ...}$$

and hence

$$\lim_{x\to 0}\sqrt{x} = \nexists\ (= \text{undefined, ...})$$

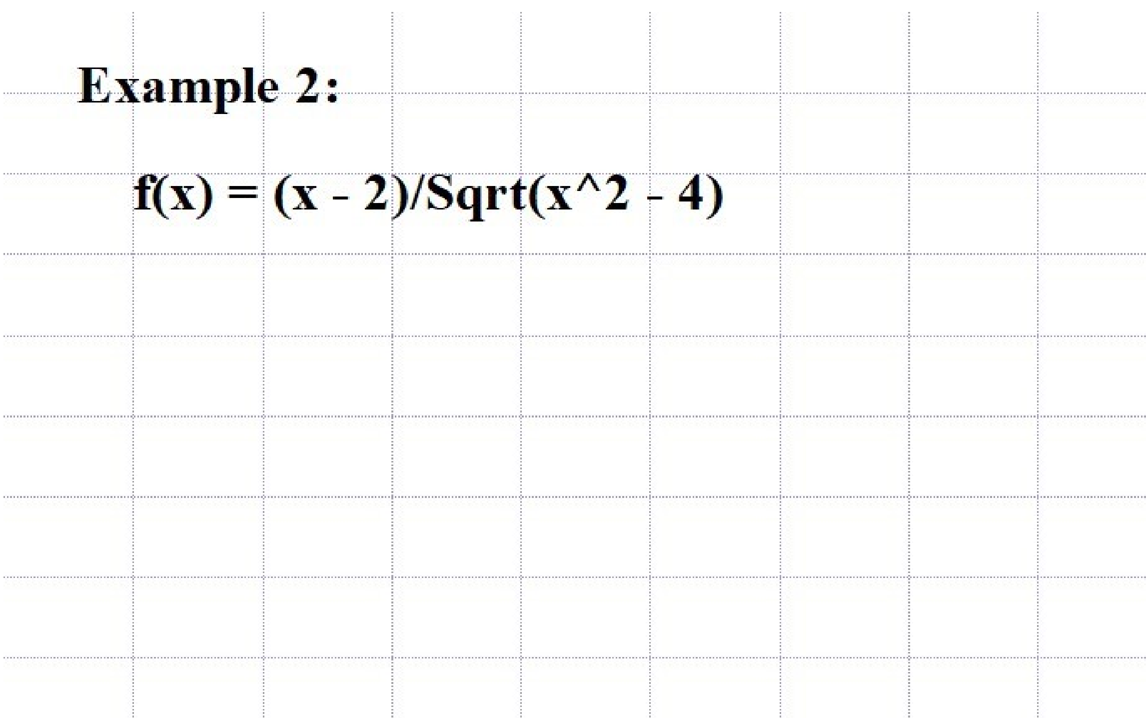
However, obviously, $\lim\limits_{x\to 0^+}\sqrt{x} = 0$



Example 02: Analyze $\lim\limits_{x\to 2}\dfrac{x-2}{\sqrt{x^2-4}}$

Solution: The domain is $(-\infty,-2)\cup(2,+\infty)$ so that $\lim\limits_{x\to 2^-}\dfrac{x-2}{\sqrt{x^2-4}}=\nexists$ and thus $\lim\limits_{x\to 2}\dfrac{x-2}{\sqrt{x^2-4}}=\nexists$. The limit from the right, however, exists:

$$
\begin{aligned}
L_{right} = \lim_{x\to 2^+}\frac{x-2}{\sqrt{x^2-4}} &= \lim_{x\to 2^+}\frac{x-2}{\sqrt{x^2-4}}\frac{\sqrt{x^2-4}}{\sqrt{x^2-4}} \ \{\text{Rationalize}\} \\
&= \lim_{x\to 2^+}\frac{(x-2)\sqrt{x^2-4}}{x^2-4} \\
&= \lim_{x\to 2^+}\frac{(x-2)\sqrt{x^2-4}}{(x-2)(x+2)} \\
&= \lim_{x\to 2^+}\frac{\sqrt{x^2-4}}{x+2} \ \{\text{Factor}\} \\
&= 0 \qquad \textbf{Converges}: \ \text{"C"}
\end{aligned}
$$



When a function has both a square root and a non-square root in the formula, we may use what I call the **Radical Trade** to evaluate the limit:

RADICAL TRADE:

$$\sqrt{\mathbf{a}^2} = |\mathbf{a}| = \begin{cases} -\mathbf{a} & \text{if } \mathbf{a} < 0 \\ 0 & \text{if } \mathbf{a} = 0 \\ +\mathbf{a} & \text{if } \mathbf{a} > 0 \end{cases} \Rightarrow \mathbf{a} = \begin{cases} -\sqrt{\mathbf{a}^2} & \text{if } \mathbf{a} < 0 \\ +\sqrt{\mathbf{a}^2} & \text{if } \mathbf{a} > 0 \end{cases}$$

Example 03: Revisit $\lim\limits_{x\to 2} f(x) = \lim\limits_{x\to 2} \dfrac{x-2}{\sqrt{x^2-4}}$

Solution:

If $x < 2 \Rightarrow x - 2 < 0$ but $f(x)$ is undefined

$$\Rightarrow \lim_{x\to 2^-} f(x) = \nexists \Rightarrow \lim_{x\to 2} f(x) = \nexists$$

If $x > 2 \Rightarrow x - 2 > 0$ and $f(x)$ is defined

$$\Rightarrow \lim_{x\to 2^+} f(x) = ?$$

Set $\mathbf{a} = x - 2 > 0 \Rightarrow x - 2 = +\sqrt{(x-2)^2}$

$$\left(\mathbf{a} = +\sqrt{\mathbf{a}^2}\right)$$

$$\text{Then } \lim_{x\to 2^+} \frac{x-2}{\sqrt{x^2-4}} = \lim_{x\to 2^+} \frac{+\sqrt{(x-2)^2}}{\sqrt{x^2-4}}$$

$$= \lim_{x\to 2^+} \sqrt{\frac{(x-2)(x-2)}{(x+2)(x-2)}}$$

$$= \lim_{x\to 2^+} \sqrt{\frac{(x-2)}{(x+2)}} \quad \{\text{Factor}\}$$

$$= 0 \qquad \textbf{Converges}: \ \text{"C"}$$

Example 2:

f(x) = (x - 2)/Sqrt(x^2 - 4)

We frequently need these

FUNdamental Facts:

1. $\frac{1}{\textbf{BIG}} = \textbf{SMALL}$
2. $\frac{1}{\textbf{SMALL}} = \textbf{BIG}$

Example 04: Find $\mathbf{L} = \underset{\mathbf{x}\to 0}{\mathbf{Lim}}\, \mathbf{x}\sqrt{4+\frac{1}{\mathbf{x}^2}}$ if it converges.

Solution: Since $\mathbf{x} < 0 \Rightarrow \mathbf{f(x)} < 0$ & $\mathbf{x} > 0 \Rightarrow \mathbf{f(x)} > 0$, wisdom dictates that we calculate one-sided limits and see if they are equal. Note that we encounter another **Indeterminate Form:** $0*(\pm\infty)$. Watch how this form is changed to determine the actual limits:

$$\mathbf{L}_{\mathbf{left}} = \underset{\mathbf{x}\to 0^-}{\mathbf{Lim}}\, \mathbf{x}\sqrt{4+\frac{1}{\mathbf{x}^2}} \Rightarrow (\approx 0)*(\approx +\infty)$$

Now $\mathbf{x} < 0 \Rightarrow \mathbf{x} = -\sqrt{\mathbf{x}^2} \Rightarrow$

$$\underset{\mathbf{x}\to 0^-}{\mathbf{Lim}}\left(-\sqrt{\mathbf{x}^2}\right)\sqrt{4+\frac{1}{\mathbf{x}^2}} = -\underset{\mathbf{x}\to 0^-}{\mathbf{Lim}}\sqrt{\mathbf{x}^2\left(4+\frac{1}{\mathbf{x}^2}\right)}$$

$$= -\underset{\mathbf{x}\to 0^-}{\mathbf{Lim}}\sqrt{4\mathbf{x}^2+1}$$

$$= \mathbf{L}_{\mathbf{left}} = -1 \quad \textbf{Converges}: \text{ "C"}$$

$$\therefore \underset{\mathbf{x}\to 0}{\mathbf{Lim}}\, \mathbf{x}\sqrt{4+\frac{1}{\mathbf{x}^2}} = \nexists \quad \textbf{Diverges}: \text{ "D"}$$

Also, $\mathrm{L}_{\mathbf{right}} = \underset{\mathbf{x}\to 0^+}{\mathbf{Lim}}\, \mathbf{x}\sqrt{4+\frac{1}{\mathbf{x}^2}} \Rightarrow (\approx 0)*(\pm\infty)$

Now $\mathbf{x} > 0 \Rightarrow \mathbf{x} = +\sqrt{\mathbf{x}^2} \Rightarrow$

$$\underset{\mathbf{x}\to 0^+}{\mathbf{Lim}}\left(\sqrt{\mathbf{x}^2}\right)\sqrt{4+\frac{1}{\mathbf{x}^2}} = +\underset{\mathbf{x}\to 0^+}{\mathbf{Lim}}\sqrt{\mathbf{x}^2\left(4+\frac{1}{\mathbf{x}^2}\right)}$$

$$= \underset{\mathbf{x}\to 0^+}{\mathbf{Lim}}\sqrt{4\mathbf{x}^2+1}$$

$$= \mathbf{L}_{\mathbf{right}} = 1 \quad \textbf{Converges}: \text{ "C"}$$

The function has a "finite jump" of two (2) units!

Example 4:

$f(x) = x \text{ Sqrt}(4 + 1/x^2)$

Example 05: Analyze $\mathop{\mathbf{Lim}}\limits_{x\to 0}\frac{|\mathbf{x}|}{\mathbf{x}}$. Note the **IF:** $\frac{0}{0}$

Solution: Again, since $\mathbf{x}<0\Rightarrow \mathbf{f}(\mathbf{x})<0\ \&\ \mathbf{x}>0\Rightarrow \mathbf{f}(\mathbf{x})>0$, we calculate one-sided limits and compare. Reminder: $|x|=\begin{cases}-\mathbf{x}\ \textbf{if}\ \mathbf{x}<0\\ 0\ \textbf{if}\ \mathbf{x}=0\\ +\mathbf{x}\ \textbf{if}\ \mathbf{x}>0\end{cases}$

We have $\mathbf{x}<0\Rightarrow|\mathbf{x}|=-\mathbf{x}\Rightarrow$

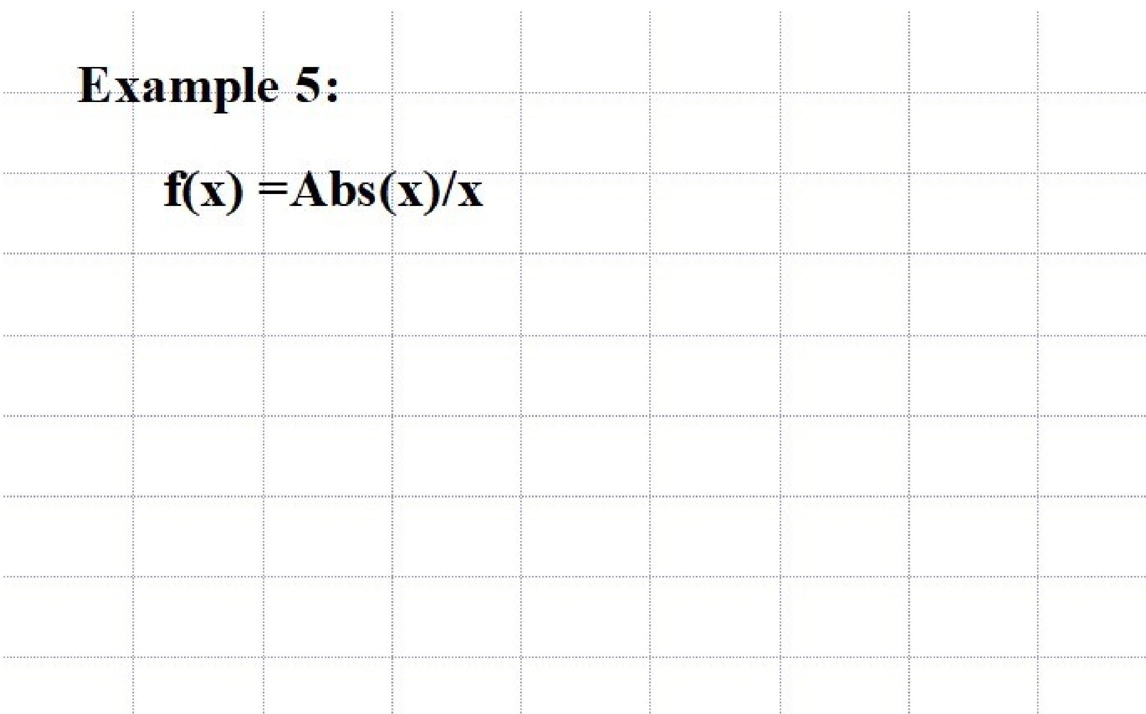
$$\mathop{\mathbf{Lim}}\limits_{x\to 0^-}\frac{|\mathbf{x}|}{\mathbf{x}}=\mathop{\mathbf{Lim}}\limits_{x\to 0^-}\frac{-\mathbf{x}}{\mathbf{x}}=\mathop{\mathbf{Lim}}\limits_{x\to 0^-}\frac{-1}{1}=\mathbf{L}_{\mathbf{left}}=-1 \quad \textbf{Converges}:\ \text{"C"}$$

$$\therefore \mathop{\mathbf{Lim}}\limits_{x\to 0}\frac{|\mathbf{x}|}{\mathbf{x}}=\nexists \quad \textbf{Diverges}:\ \text{"D"}$$

Also, $\mathbf{x}>0\Rightarrow|\mathbf{x}|=+\mathbf{x}\Rightarrow$

$$\mathop{\mathbf{Lim}}\limits_{x\to 0^+}\frac{|\mathbf{x}|}{\mathbf{x}}=\mathop{\mathbf{Lim}}\limits_{x\to 0^+}\frac{\mathbf{x}}{\mathbf{x}}=\mathop{\mathbf{Lim}}\limits_{x\to 0^+}\frac{1}{1}=\mathbf{L}_{\mathbf{right}}=1 \quad \textbf{Converges}:\ \text{"C"}$$

This function has a finite jump of two (2) units!



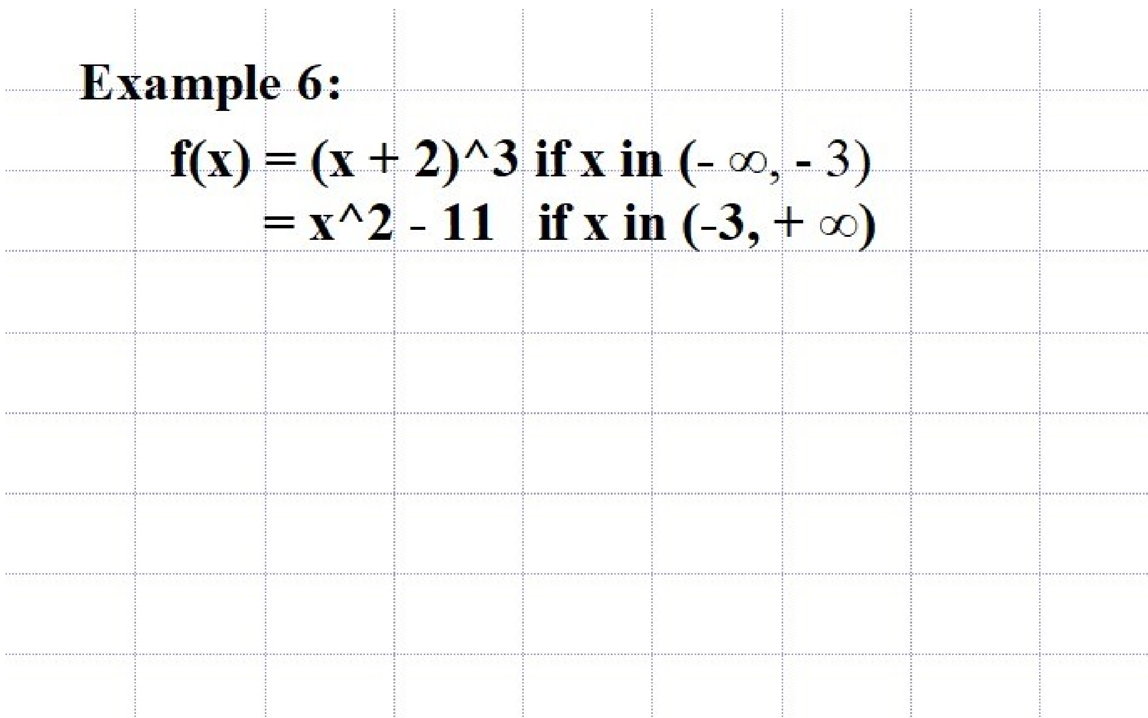
Example 06: Given $\mathbf{f}(\mathbf{x}) = \begin{cases} (\mathbf{x}+2)^3 & \textbf{if } \mathbf{x} \in (-\infty, -3) \\ \mathbf{x}^2 - 11 & \textbf{if } \mathbf{x} \in (-3, +\infty) \end{cases}$, find $\underset{\mathbf{x}\to -3}{\textbf{Lim}}\ \mathbf{f}(\mathbf{x}) \overset{?}{=} \mathbf{L}$

Solution: Calculating one-sided limits, we have

$$\underset{\mathbf{x}\to -3^-}{\textbf{Lim}}\ \mathbf{f}(\mathbf{x}) = \underset{\mathbf{x}\to -3^-}{\textbf{Lim}}\ (\mathbf{x}+2)^3 = \mathbf{L}_{\textbf{left}} = -1 \quad \textbf{Converges}:\ \text{"}\mathbf{C}\text{"}$$

$$\underset{\mathbf{x}\to -3^+}{\textbf{Lim}}\ \mathbf{f}(\mathbf{x}) = \underset{\mathbf{x}\to -3^+}{\textbf{Lim}}\ \mathbf{x}^2 - 11 = \mathbf{L}_{\textbf{right}} = -2 \quad \textbf{Converges}:\ \text{"}\mathbf{C}\text{"}$$

Therefore, $\underset{\mathbf{x}\to -3}{\textbf{Lim}}\ \mathbf{f}(\mathbf{x}) = \textbf{Undefined},\ldots$



Example 07: Given $\mathbf{f_1(x)} = \begin{cases} (\mathbf{x}+2)^3 & \textbf{if } \mathbf{x} \in (-\infty, -3) \\ \mathbf{x}^2 - 10 & \textbf{if } \mathbf{x} \in (-3, +\infty) \end{cases}$, find

$\mathbf{L} = \lim\limits_{\mathbf{x} \to -3} \mathbf{f_1(x)}$, if it converges.

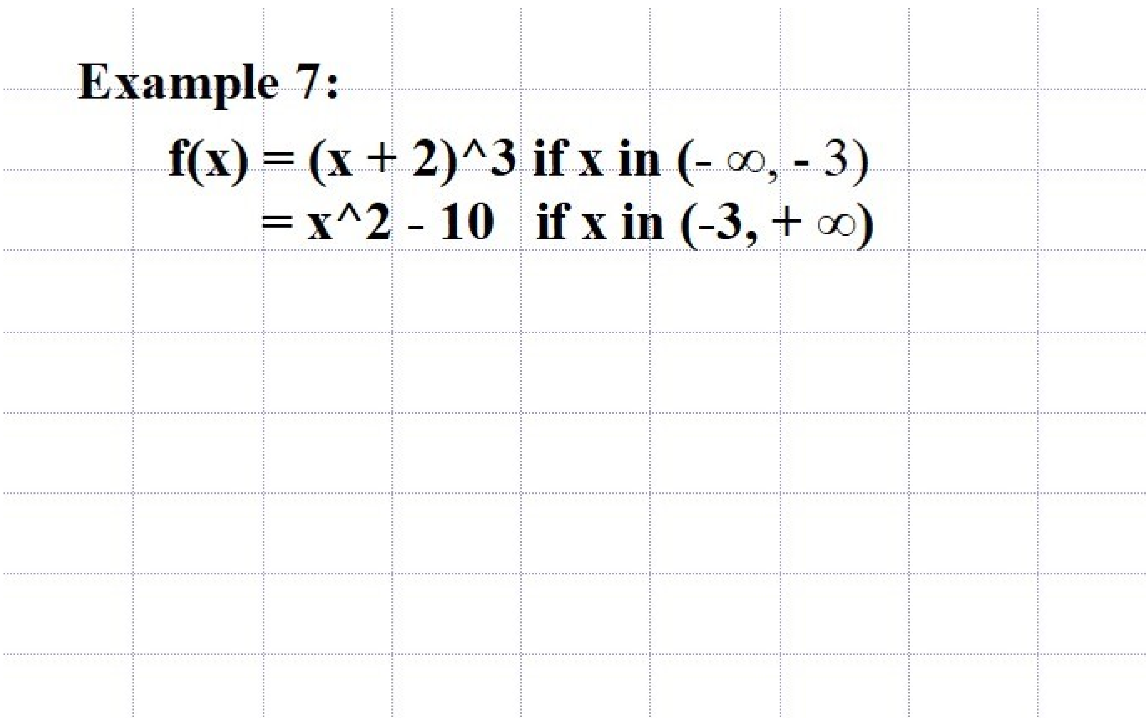
Solution: One-sided limits yield

$$\lim_{\mathbf{x} \to -3^-} \mathbf{f_1(x)} = \lim_{\mathbf{x} \to -3^-} (\mathbf{x}+2)^3 = \mathbf{L_{left}} = -1 \quad \textbf{Converges}: \ \text{"}\mathbf{C}\text{"}$$

$$\lim_{\mathbf{x} \to -3^+} \mathbf{f_1(x)} = \lim_{\mathbf{x} \to -3^+} \mathbf{x}^2 - 10 = \mathbf{L_{right}} = -1 \quad \textbf{Converges}: \ \text{"}\mathbf{C}\text{"}$$

Therefore, $\lim\limits_{\mathbf{x} \to -3} \mathbf{f_1(x)} = -1 \quad \textbf{Converges}: \ \text{"}\mathbf{C}\text{"}$

There is a hole in the graph at $(-3, -1)$!



Example 08: Given $\mathbf{f}_2(\mathbf{x}) = \begin{cases} (\mathbf{x}+2)^3 & \textbf{if } \mathbf{x} \in (-\infty, -3) \\ \mathbf{x}^2 - 10 & \textbf{if } \mathbf{x} \in (-3, +\infty) \\ \mathbf{f}_2(-3) = 5 & \end{cases}$, find

$\mathbf{L} = \underset{\mathbf{x}\to -3}{\mathbf{Lim}}\ \mathbf{f}_2(\mathbf{x})$, if it exists (converges).

Solution: We have

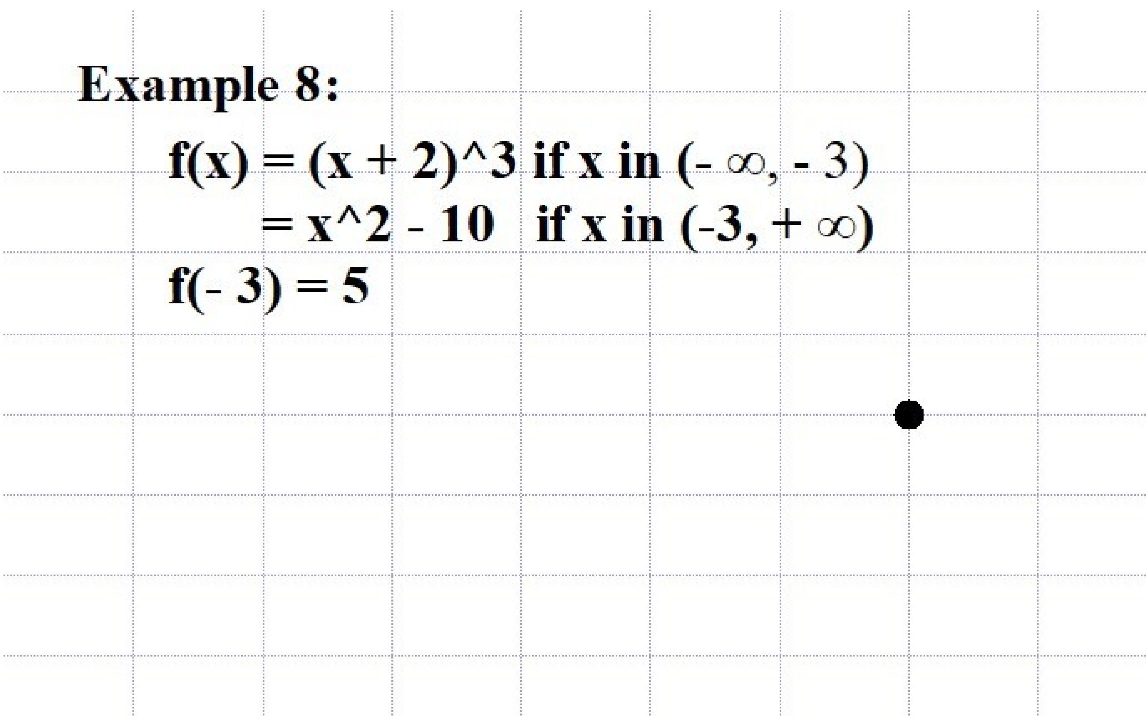
$$\underset{\mathbf{x}\to -3^-}{\mathbf{Lim}}\ \mathbf{f}_2(\mathbf{x}) = \underset{\mathbf{x}\to -3^-}{\mathbf{Lim}}\ (\mathbf{x}+2)^3 = \mathbf{L}_{\mathbf{left}} = -1 \quad \textbf{Converges}: \ \textbf{"C"}$$

$$\underset{\mathbf{x}\to -3^+}{\mathbf{Lim}}\ \mathbf{f}_2(\mathbf{x}) = \underset{\mathbf{x}\to -3^+}{\mathbf{Lim}}\ \mathbf{x}^2 - 10 = \mathbf{L}_{\mathbf{right}} = -1 \quad \textbf{Converges}: \ \textbf{"C"}$$

Therefore, $\underset{\mathbf{x}\to -3}{\mathbf{Lim}}\ \mathbf{f}_2(\mathbf{x}) = -1$

There is a hole in the graph at $(-3,-1)$ since $5 = \mathbf{f}_2(-3) \neq \mathbf{L} = -1$

Important Conclusion #1: The limit has nothing to do with whether or not the function is defined at x_0 .



Example 09: Given $\mathbf{f}_3(\mathbf{x}) = \begin{cases} (\mathbf{x}+2)^3 & \textbf{if } \mathbf{x} \in (-\infty,-3) \\ \mathbf{x}^2 - 10 & \textbf{if } \mathbf{x} \in (-3,+\infty) \\ \mathbf{f}_2(-3) = -1 & \end{cases}$, find

$\mathbf{L} = \underset{\mathbf{x}\to -3}{\mathbf{Lim}}\ \mathbf{f}_3(\mathbf{x})$, if it exists (converges).

Solution: We have

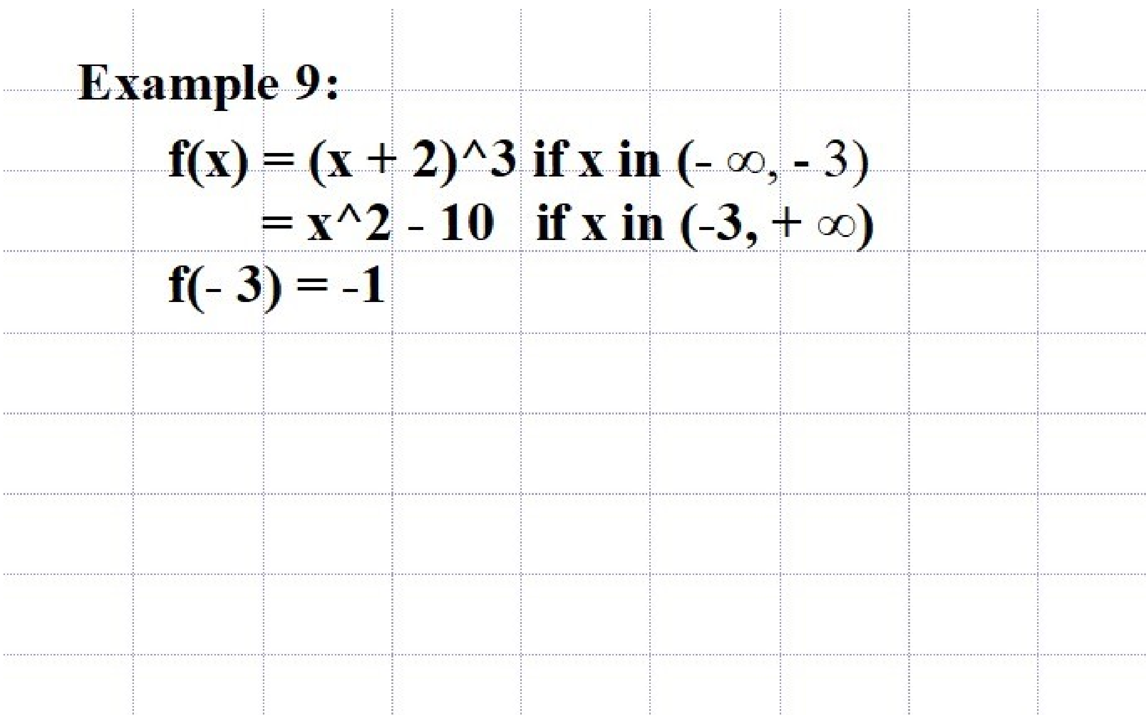
$$\underset{\mathbf{x}\to -3^-}{\mathbf{Lim}}\ \mathbf{f}_3(\mathbf{x}) = \underset{\mathbf{x}\to -3^-}{\mathbf{Lim}}\ (\mathbf{x}+2)^3 = \mathbf{L}_{\mathbf{left}} = -1 \quad \mathbf{Converges}: \ \text{"}\mathbf{C}\text{"}$$

$$\underset{\mathbf{x}\to -3^+}{\mathbf{Lim}}\ \mathbf{f}_3(\mathbf{x}) = \underset{\mathbf{x}\to -3^+}{\mathbf{Lim}}\ \mathbf{x}^2 - 10 = \mathbf{L}_{\mathbf{right}} = -1 \quad \mathbf{Converges}: \ \text{"}\mathbf{C}\text{"}$$

Therefore, $\underset{\mathbf{x}\to -3}{\mathbf{Lim}}\ \mathbf{f}_3(\mathbf{x}) = -1$

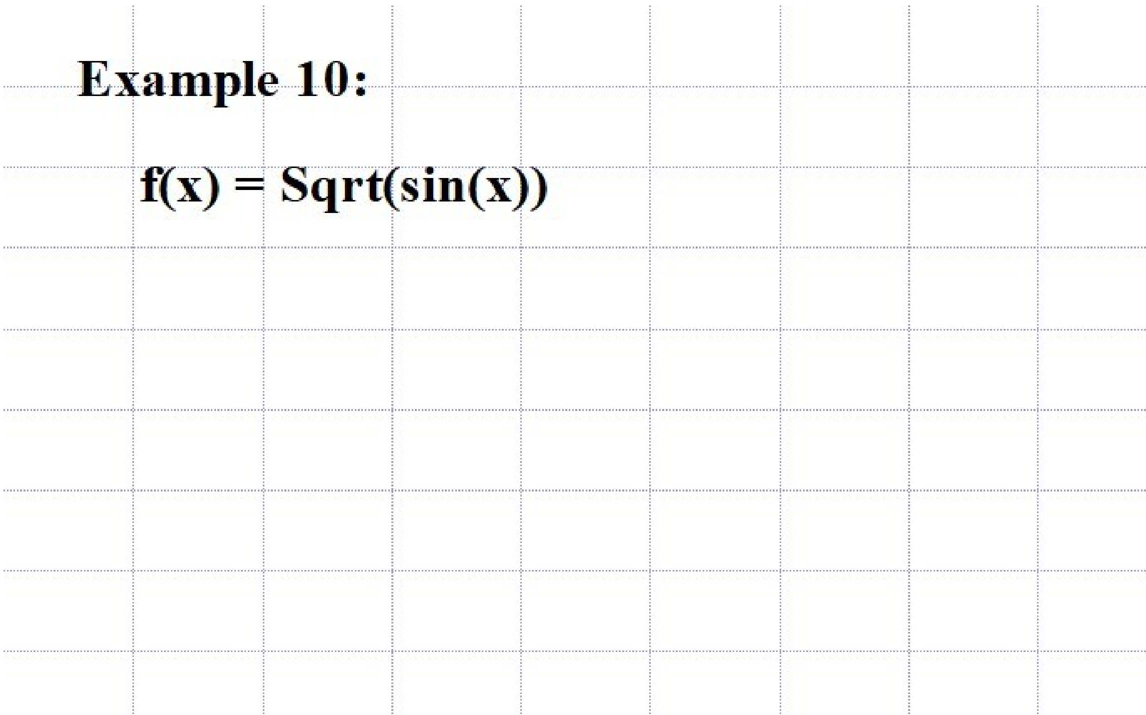
There is a NOT hole in the graph at $(-3,-1)$ since $-1 = \mathbf{f}_3(-3) = \mathbf{L} = -1$

Important Conclusion #2: If defined at x_0, the limit has nothing to do with $f(x_0)$.

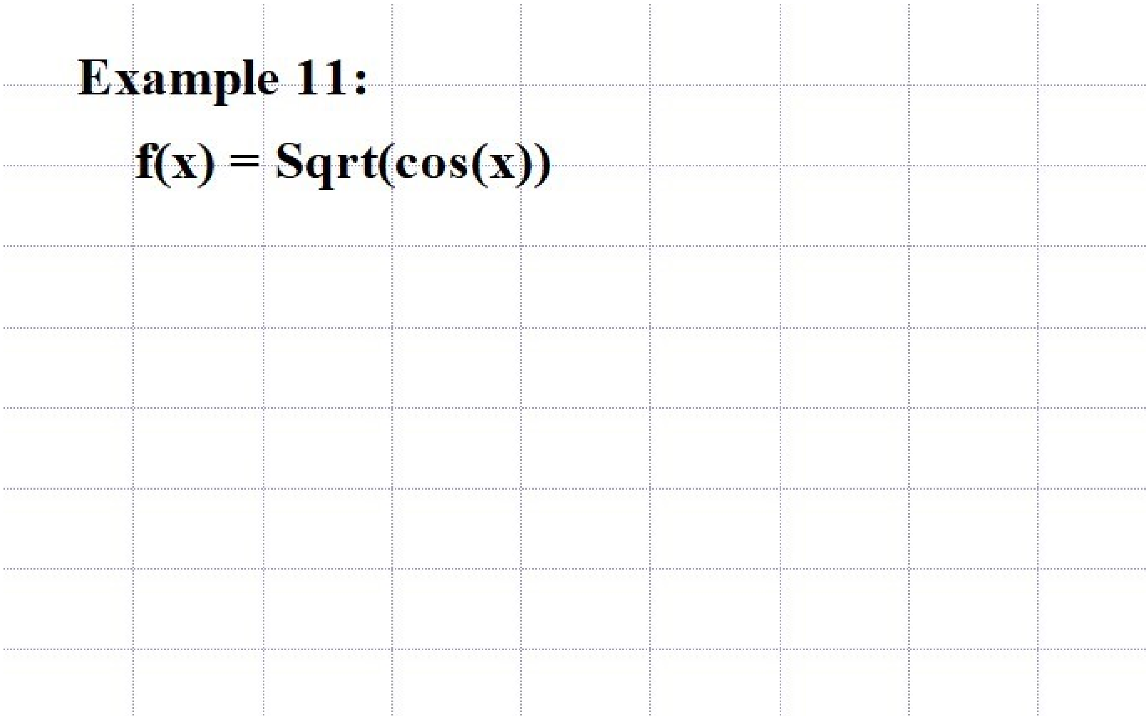


Example 10: $\lim\limits_{x\to 0}\sqrt{\sin x} = \nexists$ diverges $\{\textbf{Diverges "D"}\}$ since

$\lim\limits_{x\to 0^-}\sqrt{\sin x} = \nexists \ \{x < 0\}$ but $\lim\limits_{x\to 0^+}\sqrt{\sin x} = 0$



Example 11: $\lim\limits_{x\to 0}\sqrt{\cos x} = ? = 1$ **Converges : "C"**



Example 12: $\mathbf{\lim_{x \to 0}} \sqrt[3]{\tan \mathbf{x}} = ? = 0$ **Coverges**: "**C**"