Finite Limits at/toward Infinity

$$\lim_{x \to x_0} f(x) = L \ ; \ x_0 = \pm\infty \ ; \ L \in \mathbb{R}$$

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For a rational function **r(x)**, when $x \to \pm\infty$ $\dfrac{a_n x^n}{b_m x^m}$ dominates and we consider two (2) of the three (3) cases here:

1. $n < m \Rightarrow L = \lim\limits_{x \to x_0} r(x) = \lim\limits_{x \to x_0} \dfrac{a_n}{b_m} \dfrac{1}{x^{m-n}} = 0 \left\{ \dfrac{1}{\text{BIG}} = \text{SMALL} \right\}$

   We call the line **y = 0** a **horizontal asymptote** of **r**

2. $n = m \Rightarrow L = \lim\limits_{x \to x_0} r(x) = \lim\limits_{x \to x_0} \dfrac{a_n}{b_m} = \dfrac{a_n}{b_m}$

   We call the line $y = \dfrac{a_n}{b_m}$ a **horizontal asymptote** of **r**

3. $n > m$: Considered later

Example 01: Find $\lim\limits_{x \to -\infty} \dfrac{4x^5 - 6x^3 + 2x - 1}{2x^5 + 2x^4 - 3x^2 + 9}$

Solution:

Attempting to use the Quotient Theorem to determine the limit results is another **indeterminate form**:

$$\frac{\lim\limits_{x \to +\infty} 4x^5 - 6x^3 + 2x - 1}{\lim\limits_{x \to +\infty} 2x^5 + 2x^4 - 3x^2 + 9} \Rightarrow \frac{+\infty}{+\infty} \ \left\{ \text{IF} : \frac{\pm\infty}{\pm\infty} \right\}$$

Recall: An indeterminate form means that we have NOT done the right thing yet. The right thing to do here is to divide the numeration ***AND*** denomination by "**x** to the highest power in the rational function", in this case x^5. Using the fact that

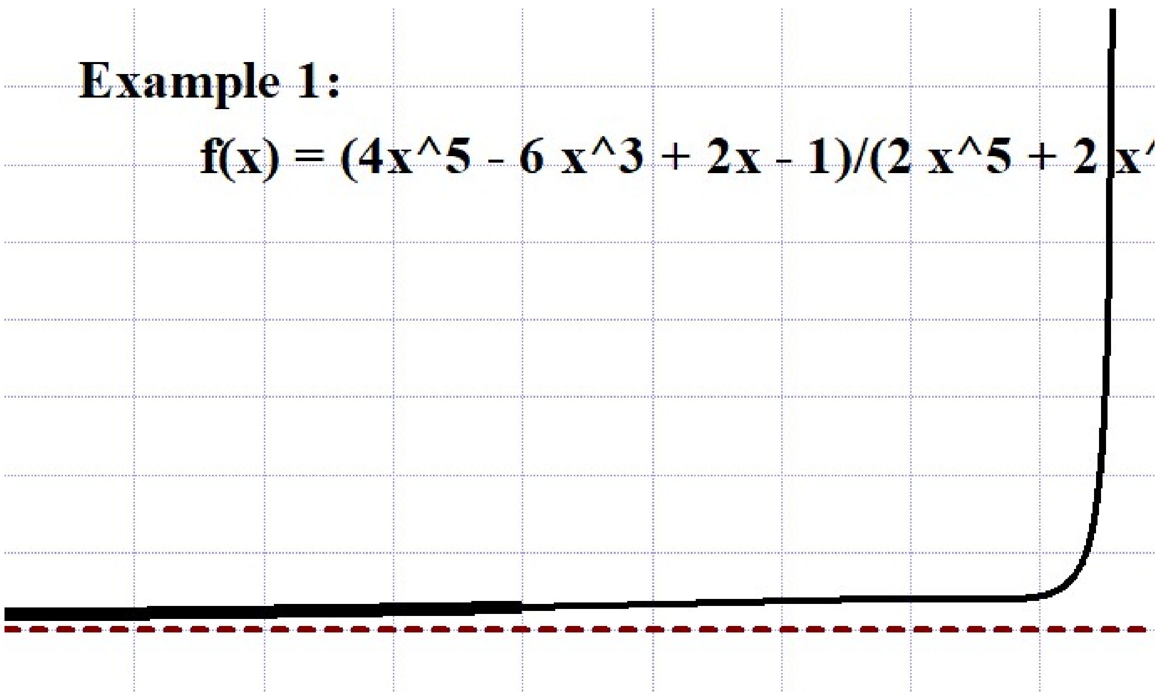
$$\frac{1}{\mathbf{BIG}} = \mathbf{SMALL}$$

yields

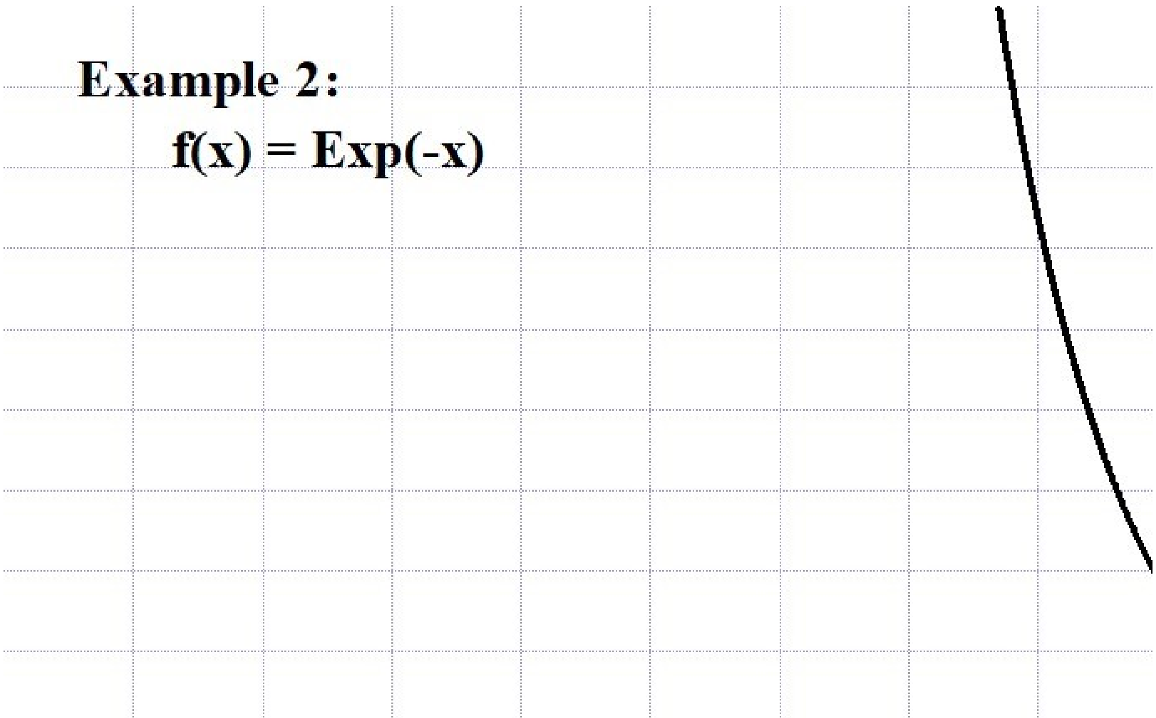
$$\lim_{\mathbf{x}\to-\infty} \frac{4\mathbf{x}^5 - 6\mathbf{x}^3 + 2\mathbf{x} - 1}{2\mathbf{x}^5 + 2\mathbf{x}^4 - 3\mathbf{x}^2 + 9} = \lim_{\mathbf{x}\to-\infty} \frac{4 - \frac{6}{\mathbf{x}^2} + \frac{2}{\mathbf{x}^4} - \frac{1}{\mathbf{x}^5}}{2 + \frac{2}{\mathbf{x}} - \frac{3}{\mathbf{x}^3} + \frac{9}{\mathbf{x}^5}}$$

$$= \frac{4}{2} \text{ Converges to } \frac{4}{2} = 2$$

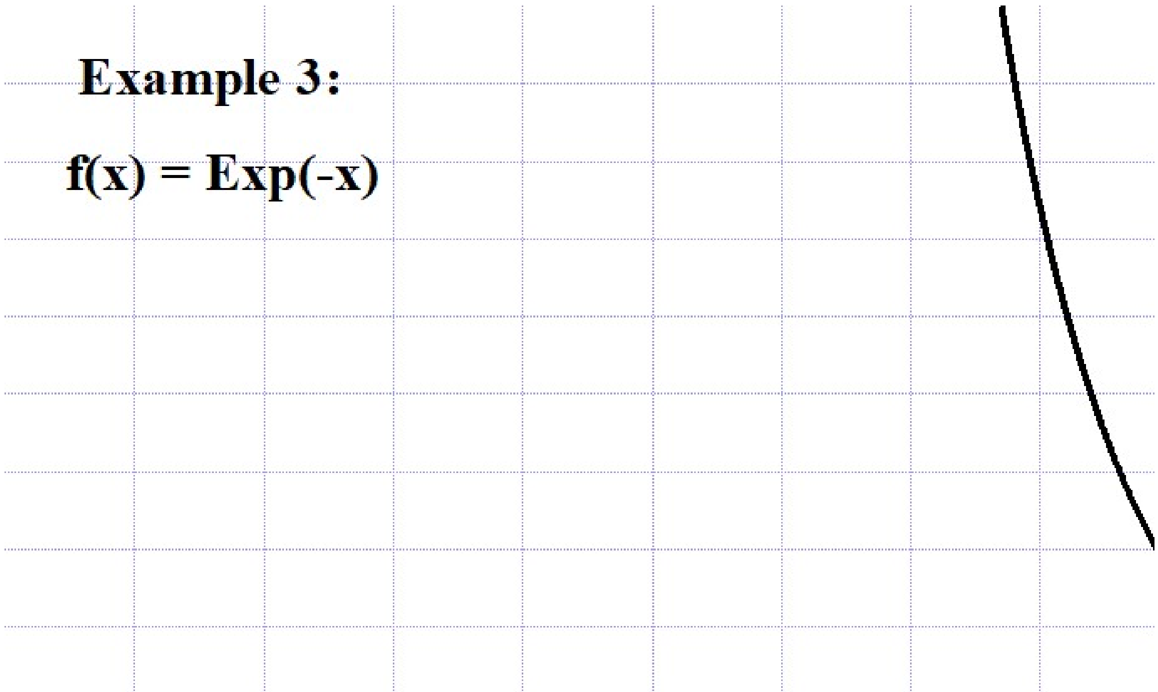
The line **y = 2** is called a horizontal asymptote of **r(x)**, as is shown in the graph below:



Example 02: $\lim\limits_{x \to -\infty} e^{-x} = +\infty$ (**Diverges to** $+\infty$) since we know the graph from PreCalculus:



Example 03: $\lim\limits_{x\to+\infty} e^{-x} = 0$ $\left(\textbf{Converges to } 0\right)$ since we know the graph from PreCalculus:



The line **y = 0** is called a horizontal asymptote.

Example 04: Find $\lim\limits_{x \to +\infty} x \sin\left(\frac{1}{x}\right)$.

Solution: First note that as $x \to +\infty,\ \frac{1}{x} \to 0(+) \Rightarrow \sin\left(\frac{1}{x}\right) \to 0(+)$. Thus the product of **x** and $\sin\left(\frac{1}{x}\right)$ yields an **indeterminate form:** $(+\infty)*(0(+))$:

$$\{\text{In general: } (0(\pm))*(\pm\infty)\}$$

We now need the substitution form of the **FUNdamental Trigonometry Limit:**

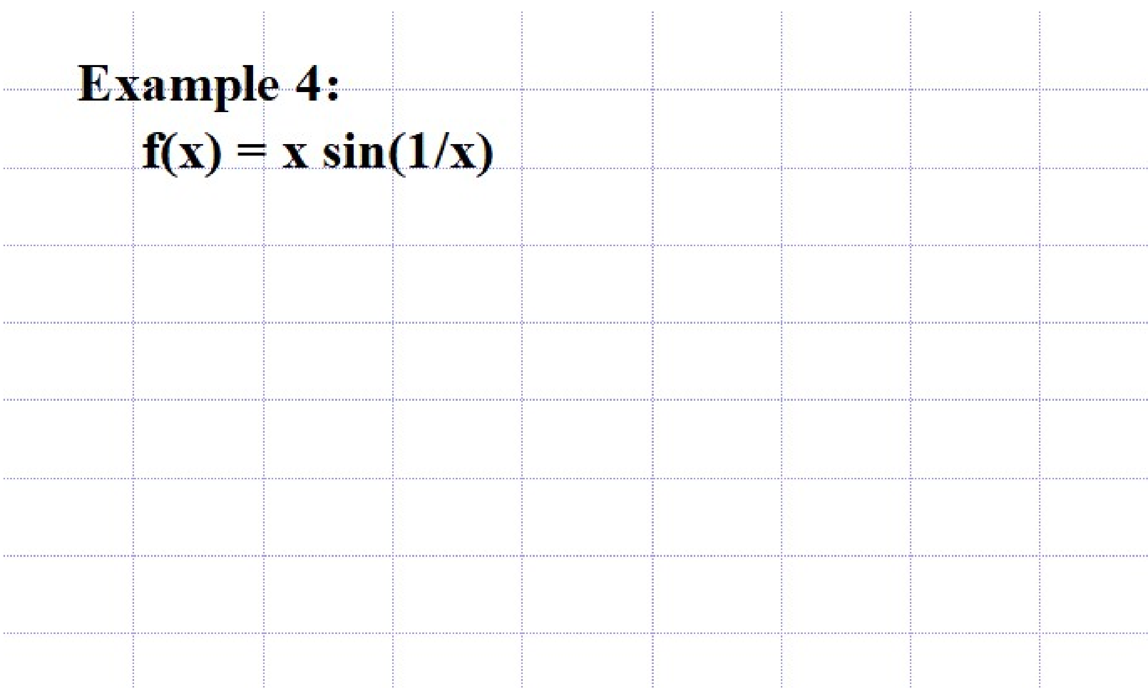
$$\lim_{u \to 0} \frac{\sin u}{u} = 1\ ;\ \lim_{u \to 0} \frac{u}{\sin u} = 1$$

We have

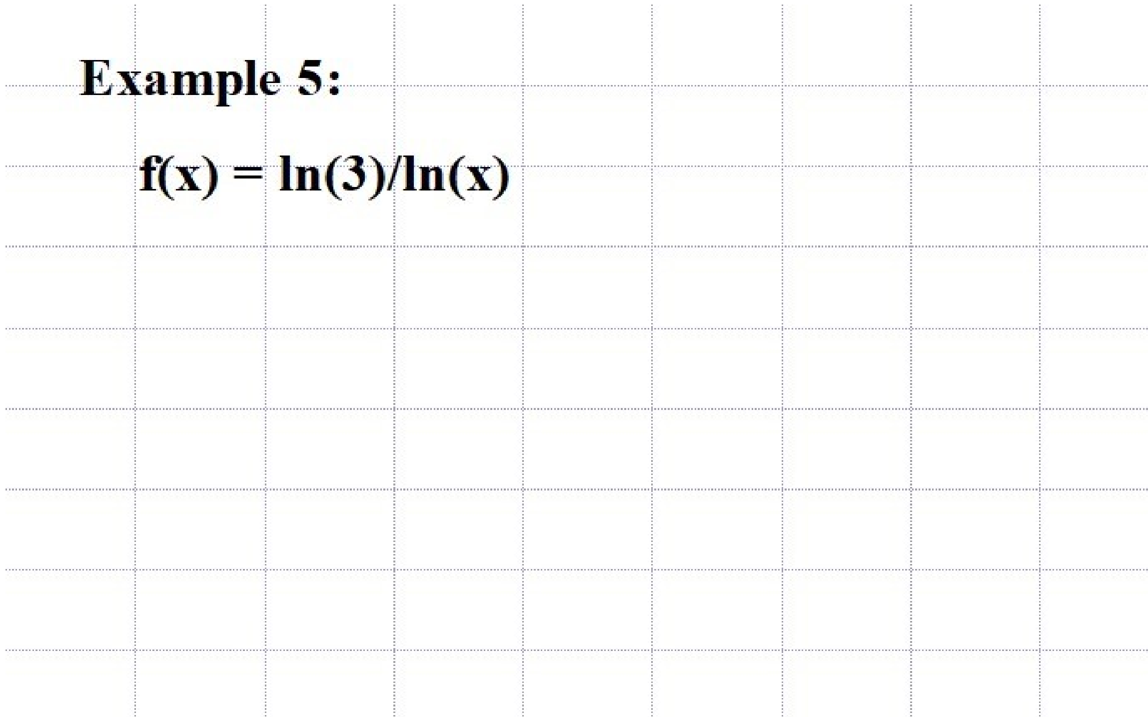
$$\text{Set } u = \frac{1}{x} \text{ so that } x \to +\infty \Rightarrow u \to 0\ ;\ x = \frac{1}{u}$$

$$\text{Thus } \lim_{x \to +\infty} x \sin\left(\frac{1}{x}\right) = \lim_{u \to 0} \frac{\sin u}{u} = 1 \quad (\textbf{Converges to } 1)$$

The line **y = 1** is a horizontal asymptote of this function whose graph is shown below:

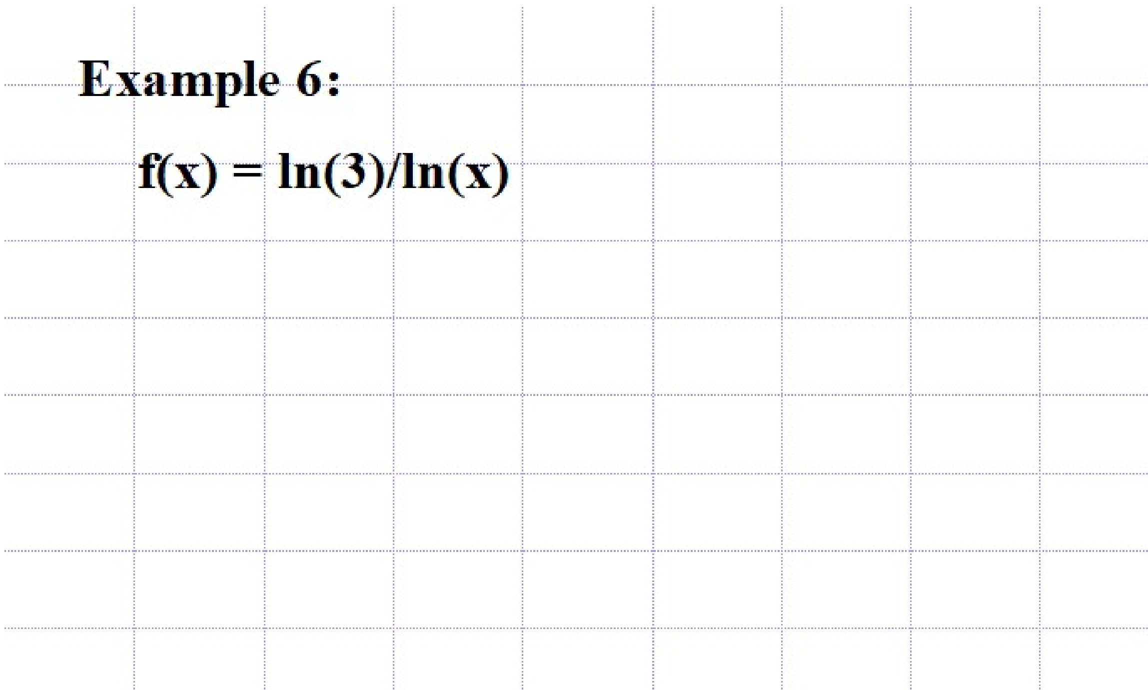


Example 05: $\displaystyle \lim_{x\to -\infty} \frac{1}{\log_3 x} = \nexists$ $\;\;$ **(Diverges)** since there are NO **y** values for x < 0:



Example 06: Find $\lim\limits_{x\to+\infty} \dfrac{1}{\log_3 x}$.

Solution: As $x \to +\infty$, we know the graph of $\log_3 x$ from PreCalculus and hence $\lim\limits_{x\to+\infty} \log_3 x = +\infty$. Hence, $\lim\limits_{x\to+\infty} \dfrac{1}{\log_3 x} = 0$ $\left(\textbf{Converges to } 0\right)$ noting $\left\{\dfrac{1}{\textbf{BIG}} = \textbf{SMALL}\right\}$**.** The line **y = 0** is a horizontal asymptote:



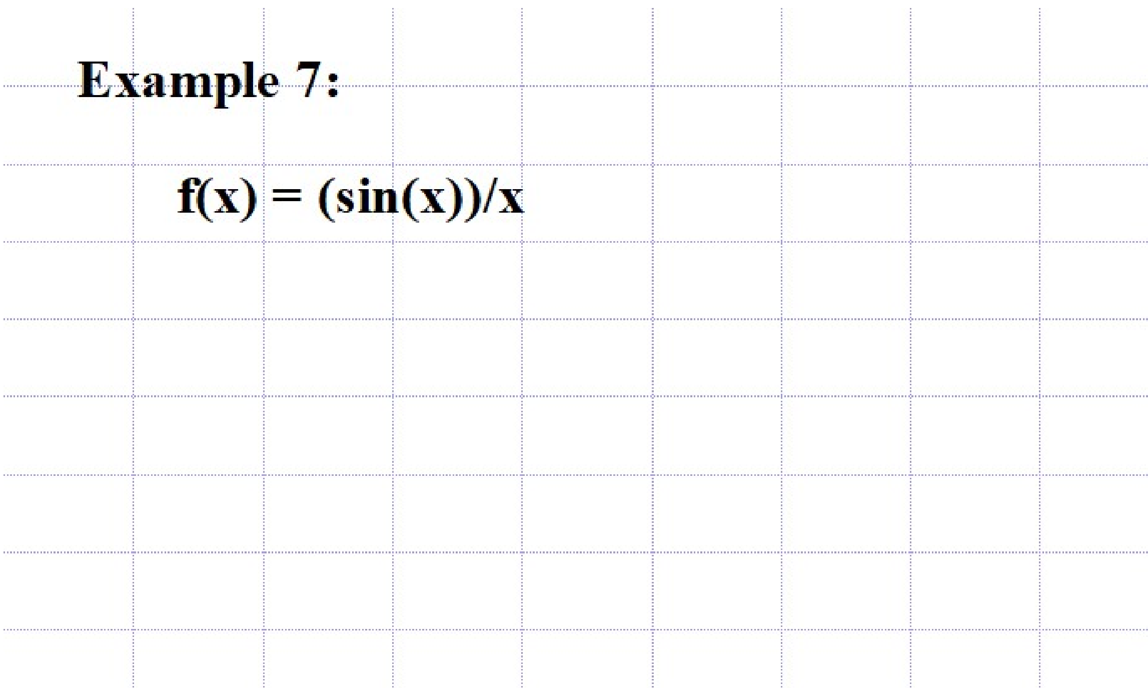
Example 07: Find $\lim\limits_{x \to +\infty} \dfrac{\sin x}{x}$.

Solution: We use the **Sandwich** (or **Squeeze) Theorem:**

$$0 \le \left|\frac{\sin x}{x}\right| = \frac{|\sin x|}{x} \le \frac{1}{x} \to 0 \text{ as } x \to +\infty$$

$$\Rightarrow \lim_{x \to +\infty} \frac{\sin x}{x} = 0$$

Since $\dfrac{\sin x}{x}$ is an even function, $\lim\limits_{x \to +\infty} \dfrac{\sin x}{x} = 0$. The line **y = 0** is a horizontal asymptote:



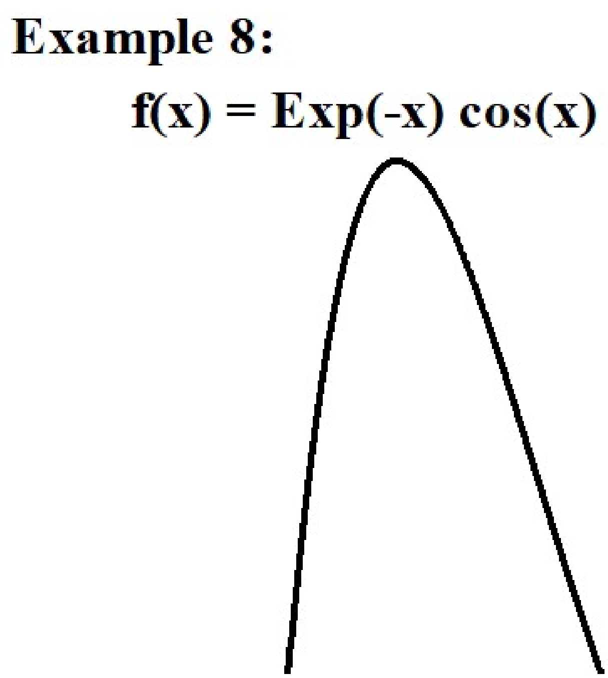
Example 08: Find $\lim\limits_{x \to -\infty} e^{-x} \cos x$.

Solution:

$\cos x$ is bounded between -1 & $+1$; $\cos x$ has an infinite number of x-intecept POINTS

Its variable amplitude: $e^{-x} \to +\infty$ as $x \to -\infty$

Therefore $\lim\limits_{x \to -\infty} e^{-x} \cos x = \nexists$ $(\textbf{Diverges})$



Example 09: Find $\lim\limits_{x\to+\infty} e^{-x}\cos x$.

Solution:

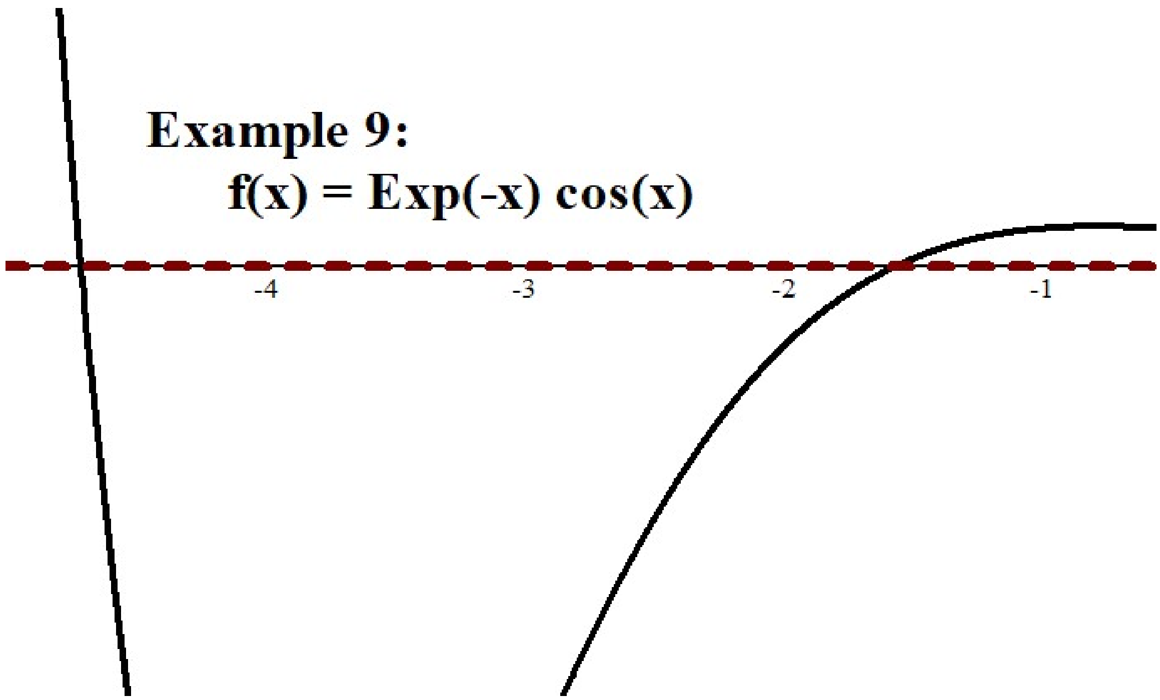
$\cos x$ is bounded between -1 & $+1$; $\cos x$ has an infinite number of x-intecept POINTS

Its variable amplitude: $e^{-x} \to 0$ as $x \to +\infty \Rightarrow$

$$0 \le \left|e^{-x}\cos x\right| \le \left|e^{-x}\right| \to 0 \text{ as } x \to +\infty \ (\textbf{SandwichTheorem})$$

Therefore $\lim\limits_{x\to+\infty} e^{-x}\cos x = 0$ $\quad$ (**Converges to** 0)

The line **y = 0** is a horizontal asymptote as the graph illustrates:



Now for the official definition of **horizontal asymptote**:

Definition: Let $\mathbf{f}$ be a function. The horizontal line $\mathbf{y} = \mathbf{y}_0$ (or $\mathbf{y} = \mathbf{y}_1$) is a **horizontal asymptote** of $\mathbf{f}$ if

1. $\displaystyle \lim_{\mathbf{x} \to -\infty} \mathbf{f}(\mathbf{x}) = \mathbf{y}_0$ or
2. $\displaystyle \lim_{\mathbf{x} \to +\infty} \mathbf{f}(\mathbf{x}) = \mathbf{y}_1$ or both are true.

There may be 0, 1, or 2 horizontal asymptotes of **f.**

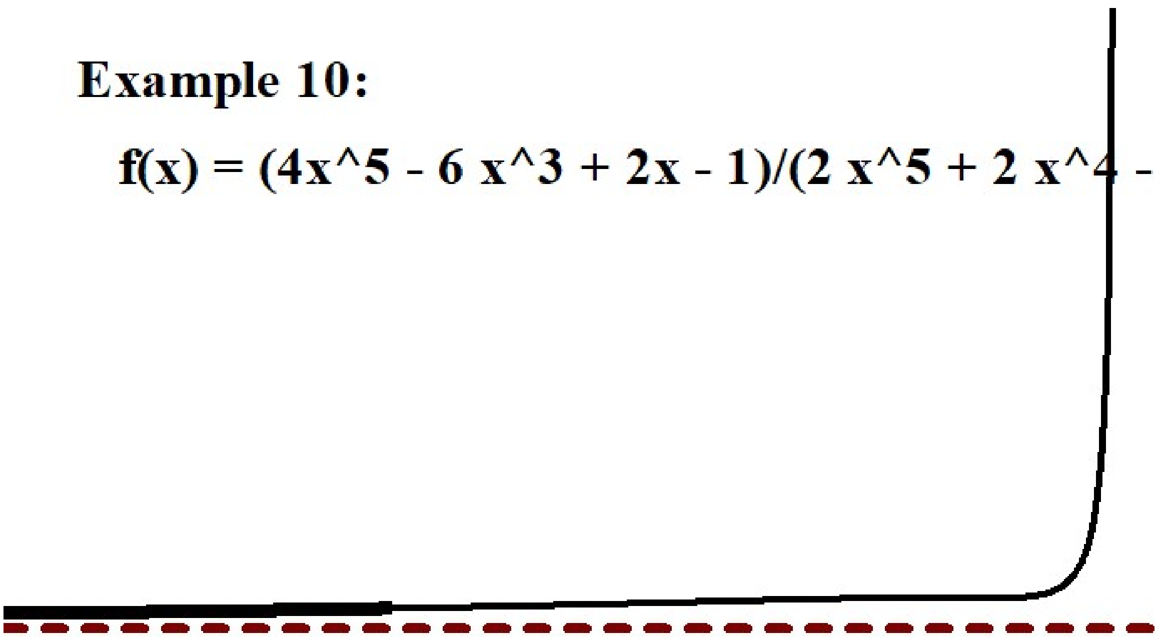
Example 10: Find $\lim\limits_{x \to -\infty} \dfrac{4x^5 - 6x^3 + 2x - 1}{2x^5 + 2x^4 - 3x^2 + 9}$.

Solution: We have , dividing numerator and denominator by x^5,

$$\lim_{x \to -\infty} \frac{4x^5 - 6x^3 + 2x - 1}{2x^5 + 2x^4 - 3x^2 + 9} = \lim_{x \to -\infty} \frac{4 - \frac{6}{x^2} + \frac{2}{x^4} - \frac{1}{x^5}}{2 + \frac{2}{x} - \frac{3}{x^3} + \frac{9}{x^5}}$$

$$= \frac{4}{2} \quad \left(\textbf{Converges to } \frac{4}{2} = 2 \right)$$

The line **y = 2** is therefore a horizontal asymptote:



Note: $\lim\limits_{x \to +\infty} \dfrac{4x^5 - 6x^3 + 2x - 1}{2x^5 + 2x^4 - 3x^2 + 9}$

Example 11: Find the horizontal asymptotes of $\mathbf{f}(\mathbf{x}) = \dfrac{4\mathbf{x}-3}{\sqrt{2\mathbf{x}^2+5\mathbf{x}-3}}$.

Solution:

First Limit: $x \to -\infty$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{4x-3}{\sqrt{2x^2+5x-3}}$$

$$= \lim_{x\to-\infty} \frac{-\sqrt{(4x-3)^2}}{\sqrt{2x^2+5x-3}} \qquad \left(4x-3<0 \Rightarrow 4x-3 = -\sqrt{(4x-3)^2}\right)$$

$$= -1 \bullet \lim_{x\to-\infty} \frac{\sqrt{16x^2-24x+9}}{\sqrt{2x^2+5x-3}}$$

$$= -1 \bullet \lim_{x\to-\infty} \sqrt{\frac{16x^2-24x+9}{2x^2+5x-3}}$$

$$= -1 \bullet \lim_{x\to-\infty} \sqrt{\frac{16-\frac{24}{x}+\frac{9}{x^2}}{2+\frac{5}{x}-\frac{3}{x^2}}}$$

$$= -2\sqrt{2} \quad (\textbf{Converges}) \Rightarrow \mathbf{y} = -2\sqrt{2} \text{ is a horizontal asymptote}$$

Note: The **"Radical Trade"** was used in evaluating the First Limit and we will use it in the Second Limit too. limit.

Second Limit: $x \to +\infty$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{4x-3}{\sqrt{2x^2+5x-3}}$$

$$= \lim_{x\to+\infty} \frac{+\sqrt{(4x-3)^2}}{\sqrt{2x^2+5x-3}} \qquad \left(4x-3>0 \Rightarrow 4x-3=\sqrt{(4x-3)^2}\right)$$

$$= \lim_{x\to+\infty} \frac{\sqrt{16x^2-24x+9}}{\sqrt{2x^2+5x-3}}$$

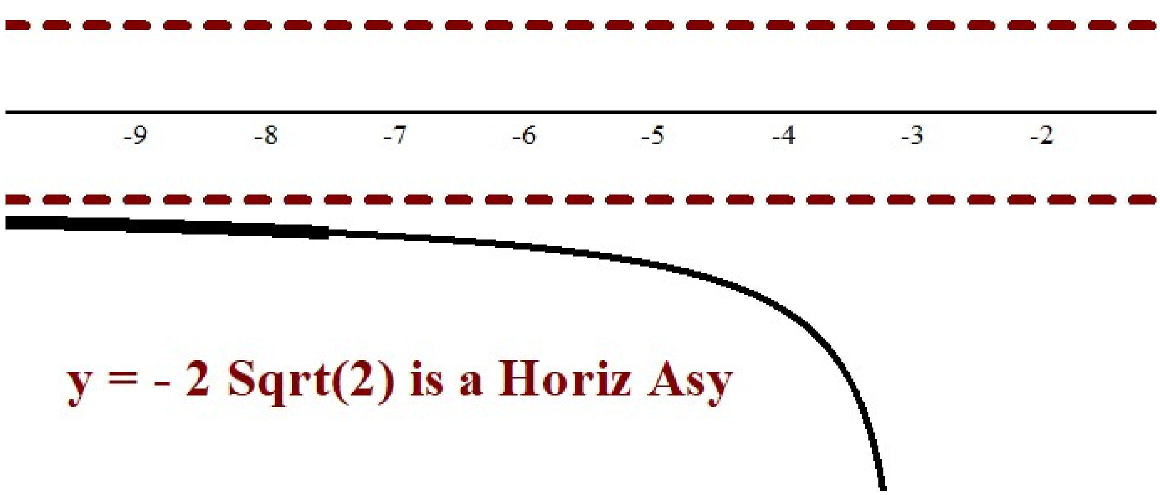
$$= \lim_{x\to+\infty} \sqrt{\frac{16x^2-24x+9}{2x^2+5x-3}}$$

$$= \lim_{x\to+\infty} \sqrt{\frac{16-\frac{24}{x}+\frac{9}{x^2}}{2+\frac{5}{x}-\frac{3}{x^2}}}$$

$$=+2\sqrt{2} \quad (\textbf{Converges}) \Rightarrow y=+2\sqrt{2} \text{ is a horizontal asymptote}$$

Example 11:

f(x) = (4x - 3)/Sqrt(2 x^2 + 5 x - 3)



Example 12: Find the horizontal asymptotes of $\mathbf{f}(\mathbf{x}) = \frac{1}{\mathbf{x}}\sqrt{4\mathbf{x}^2+1}$.

Solution:

First Limit:

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{1}{x}\sqrt{4x^2+1}$$

$$= \lim_{x\to-\infty} -\frac{\sqrt{4x^2+1}}{\sqrt{x^2}} \qquad \left(x<0 \Rightarrow x = -\sqrt{x^2}\right)$$

$$= -1 \bullet \lim_{x\to-\infty} \sqrt{\frac{4x^2+1}{x^2}}$$

$$= -1 \bullet \lim_{x\to-\infty} \sqrt{4+\frac{1}{x^2}}$$

$$= -2 \quad (\textbf{Converges}) \Rightarrow y = -2 \text{ is a horizontal asymptote}$$

Second limit:

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{1}{x}\sqrt{4x^2+1}$$

$$= \lim_{x\to+\infty} \frac{\sqrt{4x^2+1}}{\sqrt{x^2}} \qquad \left(x>0 \Rightarrow x = \sqrt{x^2}\right)$$

$$= \lim_{x\to+\infty} \sqrt{\frac{4x^2+1}{x^2}}$$

$$= \lim_{x\to+\infty} \sqrt{4+\frac{1}{x^2}}$$

$$= 2 \quad (\textbf{Converges}) \Rightarrow y = 2 \text{ is a horizontal asymptote}$$

Note: The **"Radical Trade"** was used in evaluating these limits.

Example 12:

f (x) =(1/x) *Sqrt(4 x^2 + 1)

