# Infinite Limits at/toward Infinity

$$\lim_{x \to x_0} f(x) = L \; ; \; x_0 = \pm\infty \; ; \; L = \pm\infty$$

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Now we allow $x_0 = \pm\infty$ :

$$\underset{-\infty \leftarrow x}{/////////////}\_\_\_\_\_\_ 0 \_\_\_\_\_\_\underset{x \to +\infty}{/////////////}$$

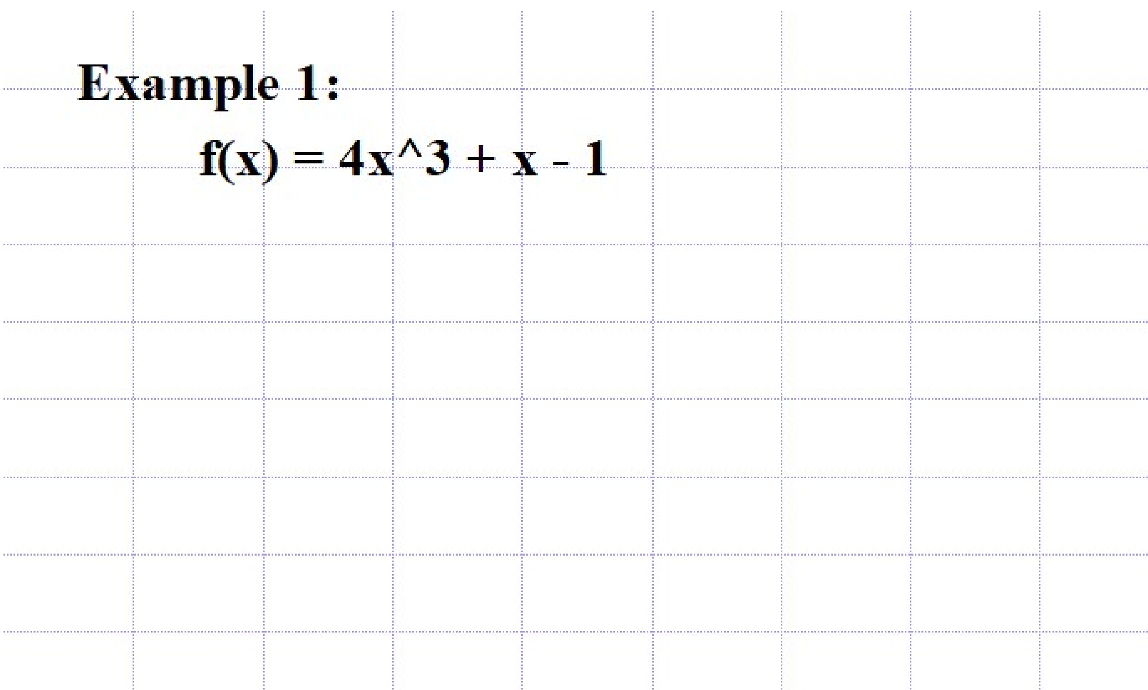
If $p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_3 x^3 + a_2 x^2 + a_1 x + a_0$ is a polynomial, then the term $a_n x^n$ dominates as $x_0 = \pm\infty$ :

**Example 01:** $\lim_{x \to -\infty} 4x^3 + x - 1 = -\infty$ $\quad$ (**Diverges to** $-\infty$) since

$x \to -\infty$

$x^3 \to -\infty$

$4x^3 \to -\infty$



The graph above also shows that $\mathbf{\underset{x\to +\infty}{Lim}}\ 4\mathbf{x}^3+\mathbf{x}-1=+\infty$ $\quad$ **(Diverges to** $+\infty$**)**

Consider a rational function:

$$\mathbf{r(x)}=\frac{\text{Polynomial \#1}}{\text{Polyomial \#2}}=\frac{\mathbf{p(x)}}{\mathbf{q(x)}}$$
$$=\frac{\mathbf{a_n x^n}+\mathbf{a_{n-1}x^{n-1}}+\ldots+\mathbf{a_3x^3}+\mathbf{a_2x^2}+\mathbf{a_1x}+\mathbf{a_0}}{\mathbf{b_m x^m}+\mathbf{b_{m-1}x^{m-1}}+\ldots+\mathbf{b_3x^3}+\mathbf{b_2x^2}+\mathbf{b_1x}+\mathbf{b_0}}$$

Assuming $\mathbf{n}>\mathbf{m}$, $\dfrac{\mathbf{a_n x^n}}{\mathbf{b_m x^m}}=\dfrac{\mathbf{a_n}}{\mathbf{b_m}}\mathbf{x}^{\mathbf{n-m}}$ dominates as $\mathbf{x}_0=\pm\infty$ :

**Example 02:** Analyze $\mathbf{\underset{x\to +\infty}{Lim}}\dfrac{3\mathbf{x}^4+6}{9\mathbf{x}^3-\mathbf{x}^2+4}$.

**Solution:**

Attempting to use the Quotient Theorem to determine the limit results is another **indeterminate form**:

$$\frac{\mathbf{\underset{x\to+\infty}{Lim}}\ 3\mathbf{x}^4+6}{\mathbf{\underset{x\to+\infty}{Lim}}\ 9\mathbf{x}^3-\mathbf{x}^2+4}\Rightarrow\frac{+\infty}{+\infty}\quad\left\{\mathbf{IF}:\frac{\pm\infty}{\pm\infty}\right\}$$

Recall: An indeterminate form means that we have NOT done the right thing yet. The right thing to do here is to divide the numeration ***AND*** denomination by "**x** to the highest power in the rational function", in this case $\mathbf{x}^4$. Using the fact that

$$\frac{1}{\mathbf{BIG}}=\mathbf{SMALL}$$

yields

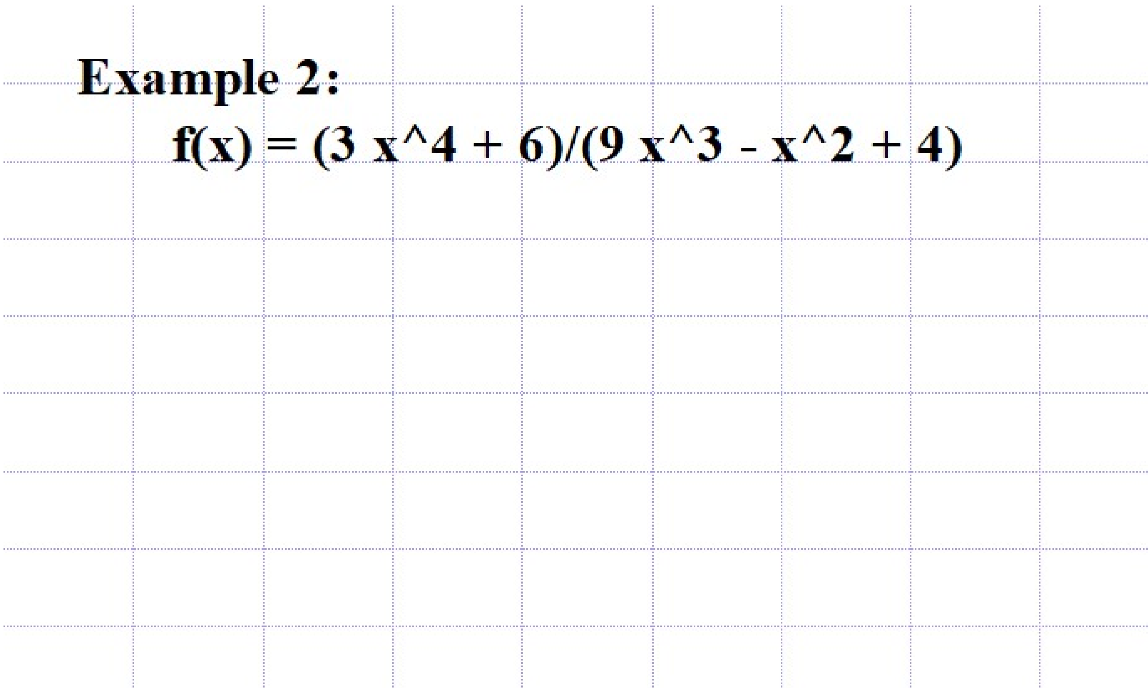
$$\mathbf{\underset{x\to+\infty}{Lim}}\frac{3\mathbf{x}^4+6}{9\mathbf{x}^3-\mathbf{x}^2+4}=\mathbf{\underset{x\to+\infty}{Lim}}\frac{\dfrac{3\mathbf{x}^4}{\mathbf{x}^4}+\dfrac{6}{\mathbf{x}^4}}{\dfrac{9\mathbf{x}^3}{\mathbf{x}^4}-\dfrac{\mathbf{x}^2}{\mathbf{x}^4}+\dfrac{4}{\mathbf{x}^4}}=\mathbf{\underset{x\to+\infty}{Lim}}\frac{3+\dfrac{6}{\mathbf{x}^4}}{\dfrac{9}{\mathbf{x}}-\dfrac{1}{\mathbf{x}^2}+\dfrac{4}{\mathbf{x}^4}}$$
$$=+\infty\quad\textbf{(Diverges to } +\infty\textbf{)}$$

Note:

1. $\frac{\neq 0}{\mathbf{x}^{\text{Power}}} \to 0$
2. $\frac{9}{\mathbf{x}} > \frac{1}{\mathbf{x}^2} > \frac{4}{\mathbf{x}^4} \Rightarrow$ denominator positive

Similarly, $\underset{\mathbf{x}\to-\infty}{\mathbf{Lim}} \frac{3\mathbf{x}^4 + 6}{9\mathbf{x}^3 - \mathbf{x}^2 + 4} = -\infty$.




Notice that in the graph above, it appears that as $\mathbf{x} \to \pm\infty$, the $\mathbf{f}$(x) values approach a line. This is in fact the case and the line we will determine is called a **slant asymptote** of **f**. To determine the equation of this line, we need to change the form again, using some PreCalculus skills. Slants asymptotes occur when $\mathbf{n} = \mathbf{m} + 1$. Using long division, we have

$$
\begin{array}{r}
\mathbf{x}/3+1/27 \\
9\mathbf{x}^3-\mathbf{x}^2+4\overline{\big)\,3\mathbf{x}^4+0\mathbf{x}^3+0\mathbf{x}^2+0\mathbf{x}+6} \\
3\mathbf{x}^4-1/3\mathbf{x}^3\qquad+4/3\mathbf{x} \\
\hline
1/3\mathbf{x}^3+0\mathbf{x}^2-4/3\mathbf{x}+6 \\
1/3\mathbf{x}^3-1/27\mathbf{x}^2+0\mathbf{x}+4/27 \\
\hline
\frac{\mathbf{x}^2}{27}-\frac{4\mathbf{x}}{3}+\frac{158}{27}
\end{array}
$$

Thus

$$\mathbf{r}(\mathbf{x})=\frac{3\mathbf{x}^4+6}{9\mathbf{x}^3-\mathbf{x}^2+4}=\overbrace{\left(\frac{9\mathbf{x}+1}{27}\right)}^{\text{Straight Line}}+\frac{\mathbf{x}^2-36\mathbf{x}+158}{27\left(9\mathbf{x}^3-\mathbf{x}^2+4\right)}$$

$$\frac{\mathbf{x}^2-36\mathbf{x}+158}{27\left(9\mathbf{x}^3-\mathbf{x}^2+4\right)}\to 0 \text{ as } x\to\pm\infty\Rightarrow$$

$$\text{Therefore } \mathbf{r}(\mathbf{x})\approx\frac{9\mathbf{x}+1}{27} \text{ as } x\to\pm\infty$$

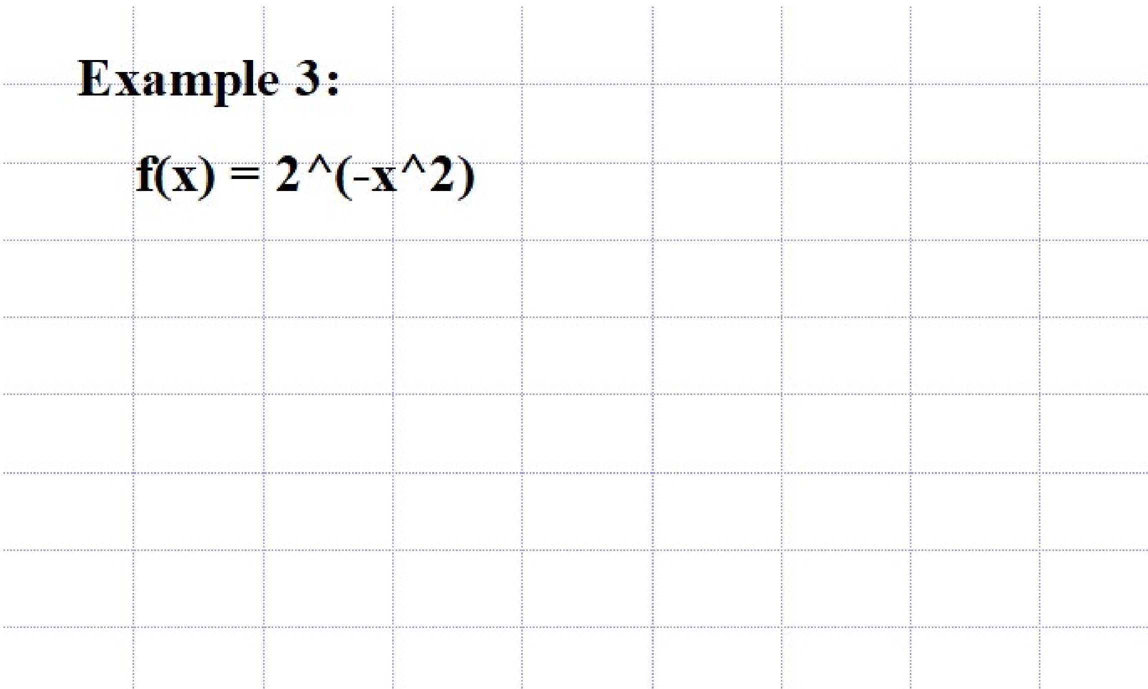
**Example 03:** $\lim\limits_{x\to -\infty} 2^{-x^2} = 0$ since

$$x \to -\infty$$

$$x^2 \to +\infty$$

$$-x^2 \to -\infty \Rightarrow 2^{-x^2} \to 0$$

Similarly $\lim\limits_{x\to +\infty} 2^{-x^2} = 0$. The line **y** = 0 is a **horizontal asymptote** as the graph below illustrates:



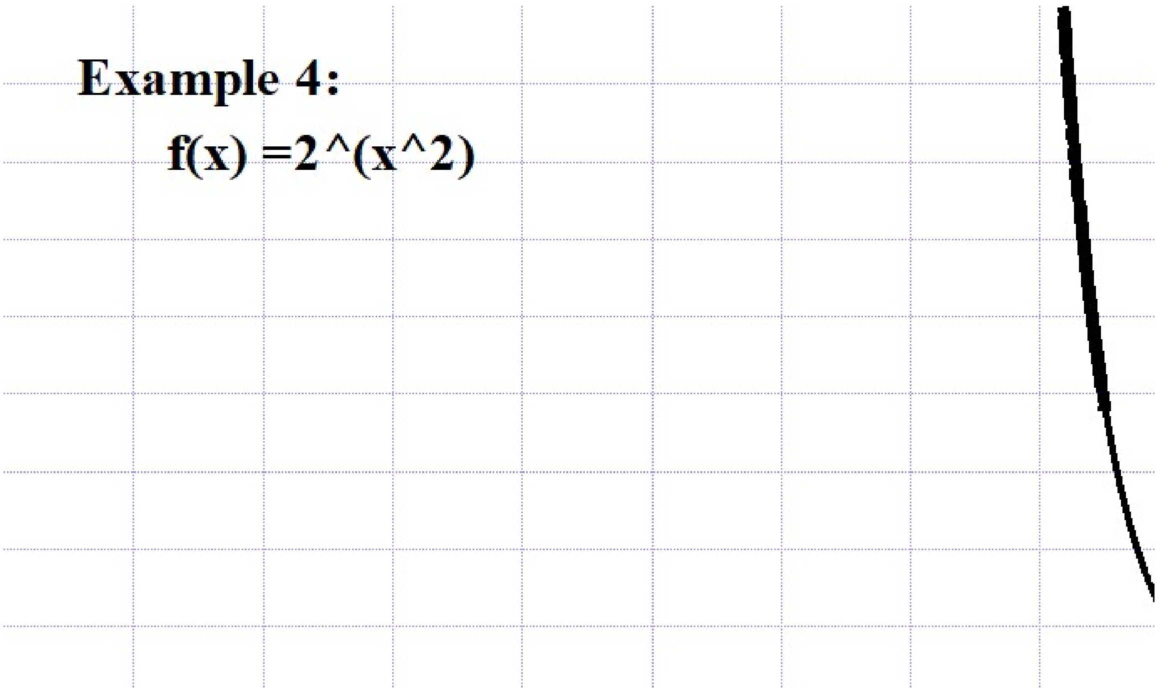
**Example 04:** $\lim\limits_{x \to -\infty} 2^{x^2} = +\infty$ $\left(\textbf{Diverges to } +\infty\right)$ since

$$x \to -\infty$$

$$x^2 \to +\infty \Rightarrow 2^{x^2} \to +\infty$$

Similarly $\lim\limits_{x \to +\infty} 2^{x^2} = +\infty$ $\left(\textbf{Diverges to } +\infty\right)$

There are NO horizontal asymptotes for this function:



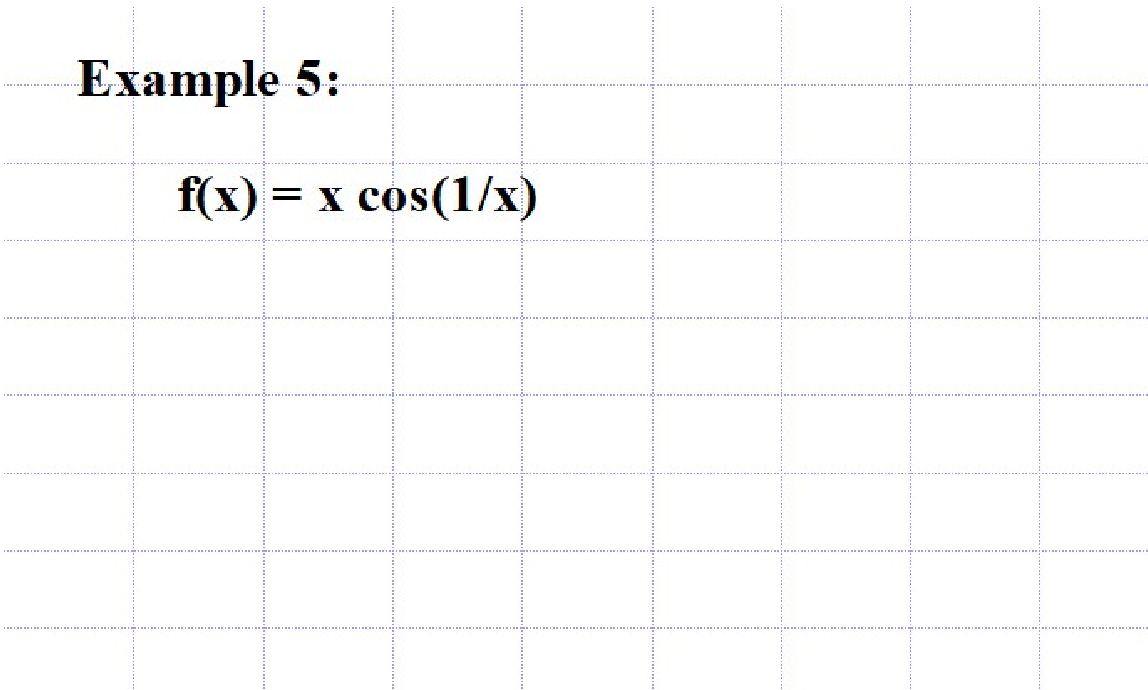

 

**Example 05:** Find $\lim\limits_{x \to +\infty} x\cos\left(\frac{1}{x}\right)$

**Solution:** First note that as $x \to +\infty$, $\frac{1}{x} \to 0(+) \Rightarrow \cos\left(\frac{1}{x}\right) \to 1$. Thus the product of **x** and $\cos\left(\frac{1}{x}\right)$ increase without bound through "+" values: $\lim\limits_{x \to +\infty} x\cos\left(\frac{1}{x}\right) = +\infty$

Similarly $\lim\limits_{x \to -\infty} x\cos\left(\frac{1}{x}\right) = -\infty$ (**Diverges to** $-\infty$)

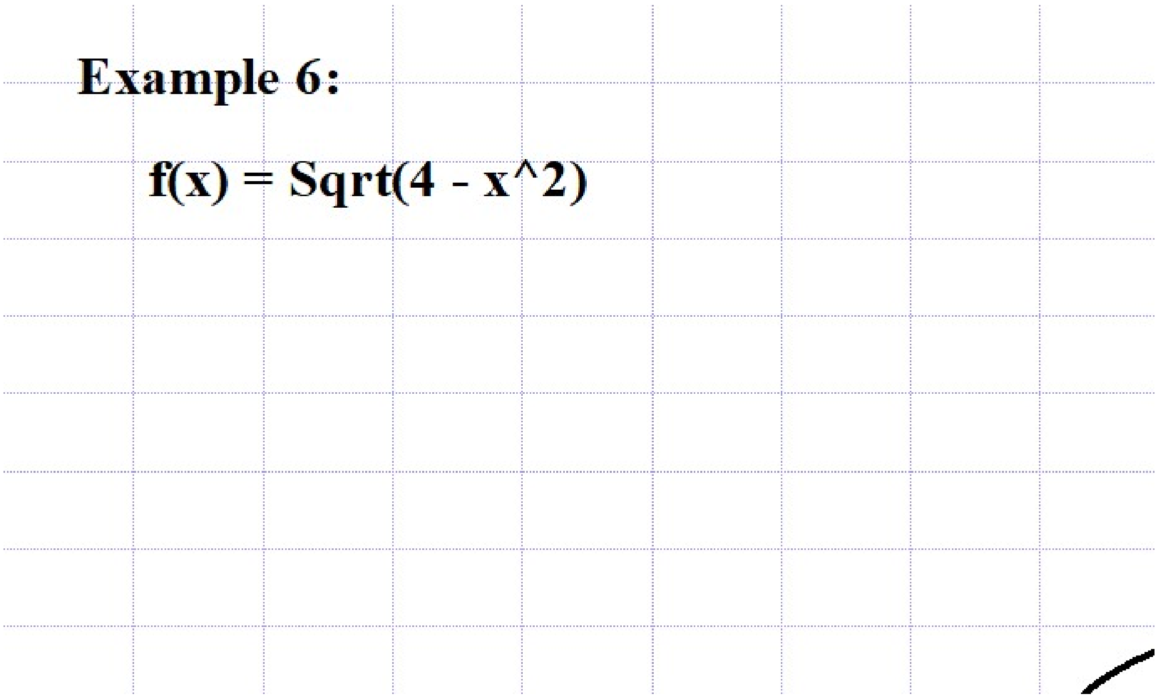
Note that the line **y** = **x** is a slant asymptote:



**Example 06:** Show $\lim\limits_{x \to -\infty} \sqrt{4 - x^2} = \nexists$ **(Diverges)**

**Solution:** Since $4 - x^2 \geq 0 \Rightarrow 4 \geq x^2 \Rightarrow x \in [-2, 2]$, as $x \to \pm\infty$ there are NO f(x) values and hence NO limit, as the graph illustrates.



**Example 07:** Why is $\lim_{x \to -\infty} \sqrt[3]{4 - x^2} = -\infty$ $\quad$ **$($Diverges to $-\infty)$**

**Solution:** Since the domain of the cube root function is *all* real numbers, we have

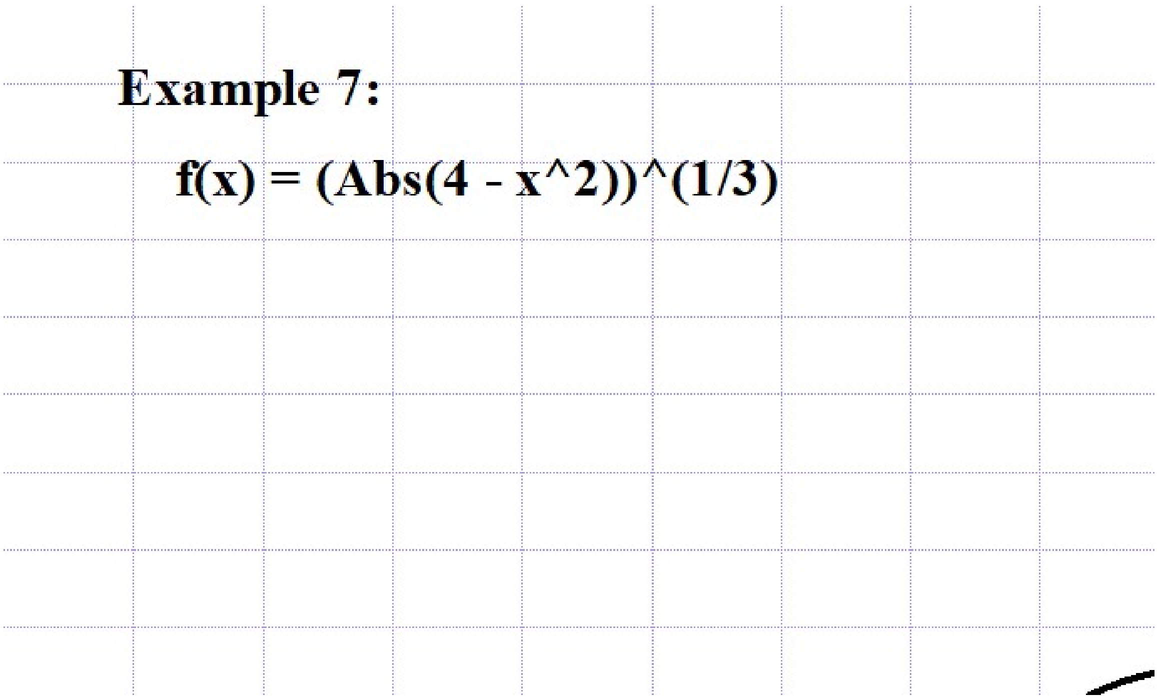
$$x \to -\infty$$

$$x^2 \to +\infty$$

$$-x^2 \to -\infty$$

$$4 - x^2 \to -\infty \Rightarrow \sqrt[3]{4 - x^2} \to -\infty$$

The graph is shown below:



**Example 8:** $\mathbf{\underset{x\to-\infty}{Lim}}\sin\mathbf{x} = \nexists\ (\textbf{Diverges})$ since we know from Trigonometry that as $\mathbf{x} \to \pm\infty$**,** the sine function oscillates between "- 1" and "+ 1". Recall that limits must be unique and that does not occur here, as the graph illustrates: