# Implicit Differentiation

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Finding the derivative $\mathbf{f}'(\mathbf{x})$ when $\mathbf{y} = \mathbf{f}(\mathbf{x})$ is given (directly) is called **explicit differentiation.** However, sometimes functions are defined **implicitly** (indirectly). For example, the circle

$$\mathbf{x}^2 + \mathbf{y}^2 = 4$$

defines two (2) functions easily found with algebra:

$$\mathbf{x}^2 + \mathbf{y}^2 = 4$$
$$\mathbf{y}^2 = 4 - \mathbf{x}^2$$
$$\mathbf{y} = \pm\sqrt{4 - \mathbf{x}^2}$$
$$\Rightarrow \begin{cases} \mathbf{f}_1(\mathbf{x}) = +\sqrt{4 - \mathbf{x}^2} & \text{(Top of circle)} \\ \mathbf{f}_2(\mathbf{x}) = -\sqrt{4 - \mathbf{x}^2} & \text{(Bottom of circle)} \end{cases}$$

**Note:** $\mathbf{y} = \mathbf{f}_1(\mathbf{x})$ satisfies $\mathbf{x}^2 + \mathbf{y}^2 = 4$:

$$\begin{aligned} \mathbf{x}^2 + \mathbf{y}^2 &= \mathbf{x}^2 + [\mathbf{f}_1(\mathbf{x})]^2 \\ &= \mathbf{x}^2 + \left[\sqrt{4 - \mathbf{x}^2}\right]^2 \\ &= \mathbf{x}^2 + 4 - \mathbf{x}^2 \\ &= 4 \end{aligned}$$

Obviously, so does $\mathbf{y} = \mathbf{f}_2(\mathbf{x})$

We can find $\mathbf{f}_1'(\mathbf{x})$ and $\mathbf{f}_2'(\mathbf{x})$ explicitly:

$$\mathbf{f}_1'(\mathbf{x}) = \frac{1}{2}\left(4 - \mathbf{x}^2\right)^{-1/2}(-2\mathbf{x}) = \frac{-\mathbf{x}}{\sqrt{4 - \mathbf{x}^2}}$$

and

$$\mathbf{f}_2'(\mathbf{x}) = \frac{\mathbf{x}}{\sqrt{4 - \mathbf{x}^2}}$$

However, if we set $\mathbf{y} = \mathbf{f}(\mathbf{x})$ and take the derivative of both sides, we obtain a formula for the derivative of *any* function satisfying the original equation:

 

$$\mathbf{x}^2 + \mathbf{y}^2 = 4 \quad \text{(TWO variables)}$$

$$\mathbf{x}^2 + [\mathbf{f}(\mathbf{x})]^2 = 4 \quad \text{(ONE variable - can use derivative formulas)}$$

$$\mathbf{D}_x\left\{\mathbf{x}^2 + [\mathbf{f}(\mathbf{x})]^2\right\} = \mathbf{D}_x[4]$$

$$2\mathbf{x} + 2[\mathbf{f}(\mathbf{x})]^1\ \mathbf{f}\,'(\mathbf{x}) = 0$$

$$\mathbf{f}\,'(\mathbf{x}) = \frac{-\mathbf{x}}{\mathbf{f}(\mathbf{x})}$$

This is called **implicit differentiation**.

**Note:** Using this *generic* **formula, we have**

$$\mathbf{f}_1\,'(\mathbf{x}) = \frac{-\mathbf{x}}{\mathbf{f}_1(\mathbf{x})} = \frac{-\mathbf{x}}{\sqrt{4-\mathbf{x}^2}}$$

and

$$\mathbf{f}_2\,'(\mathbf{x}) = \frac{-\mathbf{x}}{\mathbf{f}_2(\mathbf{x})} = \frac{\mathbf{x}}{\sqrt{4-\mathbf{x}^2}}$$

Most of the time we cannot solve the original equation for y so we *must* use implicit differentiation if we want to find derivatives.

**Example 01:** Find the slope of the tangent line (T-line) of $\mathbf{x}^2 + \mathbf{y}^2 = 4$ at $\mathsf{P}\left(\sqrt{3},1\right)$:

$$\mathbf{m}_T\Big|_{\left(\sqrt{3},1\right)}$$

**Solution:**

Two solutions:

a. **Explicit:**

Want $\mathbf{m}_T\Big|_{\left(\sqrt{3},1\right)} \Rightarrow$ need $\mathbf{f}_1(\mathbf{x}) = +\sqrt{4-\mathbf{x}^2}$ & $\left(\sqrt{3},1\right)$

$$\mathbf{m}_T\Big|_{\left(\sqrt{3},1\right)} = \mathbf{f}_1\,'(\mathbf{x})\Big|_{\sqrt{3}} = \frac{-\mathbf{x}}{\sqrt{4-\mathbf{x}^2}}\Bigg|_{\sqrt{3}} = \frac{-\sqrt{3}}{\sqrt{4-\left(\sqrt{3}\right)^2}} = -\sqrt{3}$$

It is now easy to get the equation of the T-line:

Equation of T-line:

$$\mathbf{y} - 1 = -\sqrt{3}(\mathbf{x} - \sqrt{3})$$

$$\mathbf{y} = -\sqrt{3}\mathbf{x} + 4$$

 

**b. Implicit:**

Want $\mathbf{m_T}\Big|_{(\sqrt{3},1)} \Rightarrow$ need $\mathbf{f'(x)} = \dfrac{-\mathbf{x}}{\mathbf{f(x)}}$ & $\left(\sqrt{3},1\right)$

$$\mathbf{m_T}\Big|_{(\sqrt{3},1)} = \mathbf{f'(x)}\Big|_{(\sqrt{3},1)} = \frac{-\mathbf{x}}{\mathbf{f(x)}}\Bigg|_{(\sqrt{3},1)} = \frac{-\sqrt{3}}{1} = -\sqrt{3}$$

**Example 02**: Find the slope of the T-line on the graph of $\mathbf{xy}^2 - \mathbf{x} = \mathbf{y} + 13$ at the point $(2,3)$.

**Solution**:

ASSUME that $\mathbf{y} = \mathbf{f(x)}$ satisfies $\mathbf{xy}^2 - \mathbf{x} = \mathbf{y} + 13$:

$$\mathbf{xy}^2 - \mathbf{x} = \mathbf{y} + 13$$
$$\mathbf{x}\left[\mathbf{f(x)}\right]^2 - \mathbf{x} = \mathbf{f(x)} + 13$$

Take the derivative of both sides:

$$\mathbf{D_x}\left\{\mathbf{x}\left[\mathbf{f(x)}\right]^2 - \mathbf{x}\right\} = \mathbf{D_x}\left\{\mathbf{f(x)} + 13\right\}$$
$$\mathbf{x}\left\{2\,\mathbf{f(x)}\,\mathbf{f'(x)}\right\} + \left[\mathbf{f(x)}\right]^2 - 1 = \mathbf{f'(x)}$$

Solve for $\mathbf{f'(x)}$:

$$\left[\mathbf{f(x)}\right]^2 - 1 = \mathbf{f'(x)} - \mathbf{x}\left\{2\,\mathbf{f(x)}\,\mathbf{f'(x)}\right\}$$
$$\left[\mathbf{f(x)}\right]^2 - 1 = \mathbf{f'(x)}(1 - 2\mathbf{x}\,\mathbf{f(x)})$$
$$\mathbf{f'(x)} = \frac{\left[\mathbf{f(x)}\right]^2 - 1}{1 - 2\mathbf{x}\,\mathbf{f(x)}}$$

$\mathbf{m_T}\Big|_{(2,3)} = \mathbf{f'(x)}\Big|_{(2,3)}$ (for some function - who cares which one!)

$$= \frac{\left[\mathbf{f(x)}\right]^2 - 1}{1 - 2\mathbf{x}\,\mathbf{f(x)}}\Bigg|_{(2,3)} = \frac{3^2 - 1}{1 - 2(2)(3)} = -\frac{8}{11}$$

Equation of T-line:

$$\mathbf{y} - 3 = -\frac{8}{11}\left(\mathbf{x} - 2\right)$$
$$\mathbf{y} = -\frac{8}{11}\mathbf{x} + \frac{49}{11}$$

 

**Example 03:** Find the slope of the T-line on the graph of $xy^3 - \sqrt{\mathbf{y}} = \mathbf{x}^3 + 118$ at the point $(2,4)$.

**Solution**:

ASSUME that $\mathbf{y} = \mathbf{f}(\mathbf{x})$ satisfies $xy^3 - \sqrt{\mathbf{y}} = \mathbf{x}^3 + 118$:

$$xy^3 - \sqrt{\mathbf{y}} = \mathbf{x}^3 + 118$$

$$x[\mathbf{f}(\mathbf{x})]^3 - \sqrt{\mathbf{f}(\mathbf{x})} = \mathbf{x}^3 + 118$$

$$x[\mathbf{f}(\mathbf{x})]^3 - [\mathbf{f}(\mathbf{x})]^{1/2} = \mathbf{x}^3 + 118$$

Take the derivative of both sides:

$$\mathbf{D}_x\left\{x[\mathbf{f}(\mathbf{x})]^3 - [\mathbf{f}(\mathbf{x})]^{1/2}\right\} = \mathbf{D}_x\left\{\mathbf{x}^3 + 118\right\}$$

$$\mathbf{x}\left\{3[\mathbf{f}(\mathbf{x})]^2\,\mathbf{f}'(\mathbf{x})\right\} + [\mathbf{f}(\mathbf{x})]^3 * 1 - \frac{1}{2}[\mathbf{f}(\mathbf{x})]^{-1/2}\,\mathbf{f}'(\mathbf{x}) = 3\mathbf{x}^2$$

Note: Goodbye Calculus ; Hello Algebra!

$$\mathbf{x}\left\{3[\mathbf{f}(\mathbf{x})]^2\,\mathbf{f}'(\mathbf{x})\right\} + [\mathbf{f}(\mathbf{x})]^3 * 1 - \frac{1}{2\sqrt{\mathbf{f}(\mathbf{x})}}\mathbf{f}'(\mathbf{x}) = 3\mathbf{x}^2$$

$$\mathbf{x}\left\{3[\mathbf{f}(\mathbf{x})]^2\,\mathbf{f}'(\mathbf{x})\right\} - \frac{1}{2\sqrt{\mathbf{f}(\mathbf{x})}}\mathbf{f}'(\mathbf{x}) = 3\mathbf{x}^2 - [\mathbf{f}(\mathbf{x})]^3$$

$$\mathbf{f}'(\mathbf{x})\left[3\mathbf{x}[\mathbf{f}(\mathbf{x})]^2 - \frac{1}{2\sqrt{\mathbf{f}(\mathbf{x})}}\right] = 3\mathbf{x}^2 - [\mathbf{f}(\mathbf{x})]^3$$

$$\mathbf{f}'(\mathbf{x})\left[\frac{6\mathbf{x}[\mathbf{f}(\mathbf{x})]^{5/2} - 1}{2\sqrt{\mathbf{f}(\mathbf{x})}}\right] = 3\mathbf{x}^2 - [\mathbf{f}(\mathbf{x})]^3$$

$$\mathbf{f}'(\mathbf{x}) = \frac{2\sqrt{\mathbf{f}(\mathbf{x})}\left(3\mathbf{x}^2 - [\mathbf{f}(\mathbf{x})]^3\right)}{6\mathbf{x}[\mathbf{f}(\mathbf{x})]^{5/2} - 1}$$

$$\mathbf{m}_T\Big|_{(2,4)} = \frac{2\sqrt{4}\left(3(2)^2 - [4]^3\right)}{6(2)[4]^{5/2} - 1} = -\frac{208}{383}$$

Equation of T-line:

$$y - 4 = -\frac{208}{383}(x-2)$$

$$y = -\frac{208}{383}x + \left(4 + \frac{416}{383}\right) = -\frac{208}{383}x + \frac{1948}{383}$$

 

**Example 04:** Assume $\mathbf{f}(\mathbf{x})$ satisfies $\mathrm{xe}^{\mathrm{y}} - \mathbf{y}\cos\mathbf{x} = \arctan\mathbf{y}$. Find $\mathbf{f}'(\mathbf{x})$.

**Solution**:

ASSUME that $\mathbf{f}(\mathbf{x})$ satisfies $\mathrm{xe}^{\mathrm{y}} - \mathbf{y}\cos\mathbf{x} = \arctan\mathbf{y}$:

$$\mathrm{xe}^{\mathrm{y}} - \mathbf{y}\cos\mathbf{x} = \arctan\mathbf{y}$$
$$\mathbf{xe}^{\mathbf{f}(\mathbf{x})} - \mathbf{f}(\mathbf{x})\cos\mathbf{x} = \arctan\mathbf{f}(\mathbf{x})$$

Take the derivative of both sides:

$$D_x\left\{xe^{f(x)} - f(x)\cos x\right\} = D_x\left\{\arctan f(x)\right\}$$
$$xe^{f(x)} f'(x) + e^{f(x)} * 1 - \left\{-f(x)\sin x + f'(x)\cos x\right\} = \frac{1}{1+\left[f(x)\right]^2} f'(x)$$
$$e^{f(x)} + f(x)\sin x = \frac{1}{1+\left[f(x)\right]^2} f'(x) - xe^{f(x)} f'(x) + f'(x)\cos x$$
$$e^{f(x)} + f(x)\sin x = \left[\frac{1}{1+\left[f(x)\right]^2} - xe^{f(x)} + \cos x\right] f'(x)$$
$$f'(x) = \frac{e^{f(x)} + f(x)\sin x}{\frac{1}{1+\left[f(x)\right]^2} - xe^{f(x)} + \cos x}$$

Wow! What a calculation …

 