Logarithmic Differentiation

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We will be using a few of the properties of logarithms:

1. Logarithmic Property #1: $\log_{\mathbf{b}} \square^{\triangle} = \triangle * \log_{\mathbf{b}} \square$

   Trades exponentiation for multiplication

   Note: NO property – $\left(\log_{\mathbf{b}} \square\right)^{\triangle}$
2. Logarithmic Property #2: $\log_{\mathbf{b}} \left(\square * \Delta\right) = \log_{\mathbf{b}} \square + \log_{\mathbf{b}} \Delta$

   Trades multiplication for addition
3. Logarithmic Property #3: $\log_{\mathbf{b}} \left(\dfrac{\square}{\Delta}\right) = \log_{\mathbf{b}} \square - \log_{\mathbf{b}} \Delta$

   Trades division for subtraction

Example 01:

$$\mathbf{f}(\mathbf{x}) = \ln 3\mathbf{x} \Rightarrow \mathbf{f}'(\mathbf{x}) = \frac{1}{3\mathbf{x}} * 3 = \frac{1}{\mathbf{x}} \quad \{\textbf{Chain Rule}\}$$

Also: $\mathbf{f}(\mathbf{x}) = \ln 3\mathbf{x} = \ln \mathbf{x} + \ln 3 \Rightarrow \mathbf{f}'(\mathbf{x}) = \dfrac{1}{\mathbf{x}} + 0 = \dfrac{1}{\mathbf{x}} \quad \{\textbf{Log Property}\}$

Example 02:

$$\mathbf{f}(\mathbf{x}) = \ln \mathbf{x}^3 \Rightarrow \mathbf{f}'(\mathbf{x}) = \frac{1}{\mathbf{x}^3} * 3\mathbf{x}^2 = \frac{3}{\mathbf{x}} \quad \{\textbf{Chain Rule}\}$$

Also: $\mathbf{f}(\mathbf{x}) = \ln \mathbf{x}^3 = 3\ln \mathbf{x} \Rightarrow \mathbf{f}'(\mathbf{x}) = 3 * \dfrac{1}{\mathbf{x}} \quad \{\textbf{Log Property}\}$

Example 03:

$$\mathbf{f}(\mathbf{x}) = \left(\ln \mathbf{x}\right)^3 \Rightarrow \mathbf{f}'(\mathbf{x}) = 3\left(\ln \mathbf{x}\right)^2 * \frac{1}{\mathbf{x}} = \frac{3\left(\ln \mathbf{x}\right)^2}{\mathbf{x}} \quad \{\textbf{Chain Rule}\}$$

Note: $\mathbf{f}(\mathbf{x}) = \left(\ln \mathbf{x}\right)^3 \neq 3(\ln \mathbf{x}) \quad \{\textbf{NO Log Property}\}$

Next, we are going to investigate derivatives when we have a function of the form

$h(x) = BASE^{POWER}$:

Assume: BASE $\Rightarrow$ **c**onstant or variable

Assume: POWER $\Rightarrow$ **c**onstant or variable

Temporarily, we choose constant = 2 and variable = x

We have four (4) options:

1. $\mathbf{f}(\mathbf{x}) = 2^2 = 4$ (Constant Function) $\Rightarrow \mathbf{f}'(\mathbf{x}) = 0$

2. $\mathbf{f}(\mathbf{x}) = \mathbf{x}^2$ (Power Function) $\Rightarrow \mathbf{f}'(\mathbf{x}) = 2\mathbf{x}$

3. $\mathbf{f}(\mathbf{x}) = 2^{\mathbf{x}}$ (Exponential Function) $\Rightarrow \mathbf{f}'(\mathbf{x}) = 2^{\mathbf{x}} \ln 2$

4. $\mathbf{f}(\mathbf{x}) = \mathbf{x}^{\mathbf{x}}$ (Doda Function - I named it) $\Rightarrow \mathbf{f}'(\mathbf{x}) = ?????$

   Note: Base AND Exponent both variable - no formulas to help us!

We first get a general formula for $\mathbf{h}'(\mathbf{x})$ when $\mathbf{h}(\mathbf{x}) = [\mathbf{f}(\mathbf{x})]^{\mathbf{g}(\mathbf{x})}$ - Base AND Exponent Variable :

Key: Take the "ln" of both sides :

$\ln[\mathbf{h}(\mathbf{x})] = \ln\left[[\mathbf{f}(\mathbf{x})]^{\mathbf{g}(\mathbf{x})}\right] = \mathbf{g}(\mathbf{x}) \ln[\mathbf{f}(\mathbf{x})]$ {Can NOW use the Product Rule!}

Take $\mathbf{D}_{\mathbf{x}}$ of both sides - specific & generic formulas needed:

$$\frac{\mathbf{h}'(\mathbf{x})}{\mathbf{h}(\mathbf{x})} = \mathbf{g}(\mathbf{x}) \frac{\mathbf{f}'(\mathbf{x})}{\mathbf{f}(\mathbf{x})} + \mathbf{g}'(\mathbf{x}) \ln[\mathbf{f}(\mathbf{x})]$$

$$\mathbf{h}'(\mathbf{x}) = \mathbf{h}(\mathbf{x}) \left[\mathbf{g}(\mathbf{x}) \frac{\mathbf{f}'(\mathbf{x})}{\mathbf{f}(\mathbf{x})} + \ln[\mathbf{f}(\mathbf{x})] \mathbf{g}'(\mathbf{x}) \right]$$

If you understand this procedure, you do not have to memorize this formula.

Example 04:

$\mathbf{f}\left(\mathbf{x}\right)=\mathbf{x}^{\mathbf{x}}$ (Doda Function) $\Rightarrow \mathbf{f}'(\mathbf{x}) = ?????$

Take the "ln" of both sides of the equation: $\ln\left[\mathbf{f}\left(\mathbf{x}\right)\right]=\ln\mathbf{x}^{\mathbf{x}}\overset{\text{GOOD NEWS!}}{=}\mathbf{x}*\ln\mathbf{x}$

Take derivative of both sides wrt "x"

$$\mathbf{D}_{\mathbf{x}}\left\{\ln\left[\mathbf{f}\left(\mathbf{x}\right)\right]\right\}=\mathbf{D}_{\mathbf{x}}\left\{\mathbf{x}*\ln\mathbf{x}\right\}$$

$$\frac{\mathbf{f}'(\mathbf{x})}{\mathbf{f}(\mathbf{x})}=\mathbf{x}*\frac{1}{\mathbf{x}}+\ln\mathbf{x}$$

$$\mathbf{f}'(\mathbf{x})=\mathbf{f}(\mathbf{x})*\left(\mathbf{x}*\frac{1}{\mathbf{x}}+\ln\mathbf{x}\right)$$

$$\mathbf{f}'(\mathbf{x})=\mathbf{x}^{\mathbf{x}}*\left(\mathbf{x}*\frac{1}{\mathbf{x}}+\ln\mathbf{x}\right)$$

$$\mathbf{f}'(\mathbf{x})=\mathbf{x}^{\mathbf{x}}*\left(1+\ln\mathbf{x}\right)$$

Example 05:

$\mathbf{f}(\mathbf{x})=\left[\ln\mathbf{x}\right]^{\sin\mathbf{x}}\Rightarrow\mathbf{f}'(\mathbf{x})=?$

Take the" ln" of both sides of the equation: $\ln\left[\mathbf{f}(\mathbf{x})\right]=\ln\left[\ln\mathbf{x}\right]^{\sin\mathbf{x}}=\sin\mathbf{x}*\ln\left[\ln\mathbf{x}\right]$

$$\mathbf{D}_{\mathbf{x}}\left[\ln\left[\mathbf{f}(\mathbf{x})\right]\right]=\mathbf{D}_{\mathbf{x}}\left[\sin\mathbf{x}*\ln\left[\ln\mathbf{x}\right]\right]$$

$$\frac{\mathbf{f}'(\mathbf{x})}{\mathbf{f}(\mathbf{x})}=\sin\mathbf{x}*\frac{1}{\mathbf{x}*\ln\mathbf{x}}+\ln\left[\ln\mathbf{x}\right]*\cos\mathbf{x}$$

$$\mathbf{f}'(\mathbf{x})=\left[\ln\mathbf{x}\right]^{\sin\mathbf{x}}*\left(\sin\mathbf{x}*\frac{1}{\mathbf{x}*\ln\mathbf{x}}+\ln\left[\ln\mathbf{x}\right]*\cos\mathbf{x}\right)=\left[\ln\mathbf{x}\right]^{\sin\mathbf{x}}*\left(\frac{\sin\mathbf{x}}{\mathbf{x}*\ln\mathbf{x}}+\ln\left[\ln\mathbf{x}\right]*\cos\mathbf{x}\right)$$

In **Example 06** and **Example 07**, it's NOT mandatory to take the "ln" of both sides. You should work them just using our past formulas and verify the answers are equivalent to what we get below.

Example 06:

$$\mathbf{f}(\mathbf{x}) = \frac{\mathbf{x}^2}{\sqrt{4-\mathbf{x}^2}} = \frac{\mathbf{x}^2}{\left(4-\mathbf{x}^2\right)^{1/2}} \Rightarrow \mathbf{f}'(\mathbf{x}) = ?$$

$$\text{Take the "ln" of both sides: } \ln\left[\text{f(x)}\right] = \ln\left[\frac{\mathbf{x}^2}{\left(4-\mathbf{x}^2\right)^{1/2}}\right]$$

$$= \ln\left[\mathbf{x}^2\right] - \ln\left[\left(4-\mathbf{x}^2\right)^{1/2}\right]$$

$$= 2\ln\left[\mathbf{x}\right] - \frac{1}{2}\ln\left[4-\mathbf{x}^2\right]$$

Take the $\mathbf{D}_\mathbf{x}$ of both sides:

$$\frac{\mathbf{f}'(\mathbf{x})}{\mathbf{f}(\mathbf{x})} = \frac{2}{\mathbf{x}} - \frac{1}{2}\frac{\left(-2\mathbf{x}\right)}{4-\mathbf{x}^2}$$

$$\mathbf{f}'(\mathbf{x}) = \mathbf{f}(\mathbf{x})\left[\frac{2}{\mathbf{x}} + \frac{\mathbf{x}}{4-\mathbf{x}^2}\right] = \frac{\mathbf{x}^2}{\sqrt{4-\mathbf{x}^2}}\left[\frac{2}{\mathbf{x}} + \frac{\mathbf{x}}{4-\mathbf{x}^2}\right] = \frac{\mathbf{x}^2}{\sqrt{4-\mathbf{x}^2}}\left[\frac{2\left(4-\mathbf{x}^2\right)+\mathbf{x}^2}{\mathbf{x}(4-\mathbf{x}^2)}\right]$$

$$= \frac{\mathbf{x}}{\sqrt{4-\mathbf{x}^2}}\left[\frac{8-\mathbf{x}^2}{(4-\mathbf{x}^2)}\right] = \frac{\mathbf{x}\left(8-\mathbf{x}^2\right)}{\sqrt{\left(4-\mathbf{x}^2\right)^3}}$$

Example 07:

$$\mathbf{f}(\mathbf{x}) = \frac{\left(\mathbf{x}^2 - 1\right)^2 \left(\mathbf{x}+2\right)^3}{\left(\mathbf{x}^4 - 81\right)^4} \Rightarrow \mathbf{f}'(\mathbf{x}) = ?$$

Take the "ln" of both sides: $\ln\left[f(x)\right] = \ln\left[\frac{\left(\mathbf{x}^2 - 1\right)^2 \left(\mathbf{x}+2\right)^3}{\left(\mathbf{x}^4 - 81\right)^4}\right]$

$$= \ln\left[\left(\mathbf{x}^2 - 1\right)^2\right] + \ln\left[\left(\mathbf{x}+2\right)^3\right] - \ln\left[\left(\mathbf{x}^4 - 81\right)^4\right]$$

$$= 2\ln\left(\mathbf{x}^2 - 1\right) + 3\ln\left(\mathbf{x}+2\right) - 4\ln\left(\mathbf{x}^4 - 81\right)$$

Take the $\mathbf{D}_{\mathbf{x}}$ of both sides:

$$\frac{\mathbf{f}'(\mathbf{x})}{\mathbf{f}(\mathbf{x})} = 2\frac{2\mathbf{x}}{\mathbf{x}^2 - 1} + 3\frac{1}{\mathbf{x}+2} - 4\frac{4\mathbf{x}^3}{\mathbf{x}^4 - 81}$$

$$\mathbf{f}'(\mathbf{x}) = \mathbf{f}(\mathbf{x})\left[\frac{4\mathbf{x}}{\mathbf{x}^2 - 1} + \frac{3}{\mathbf{x}+2} - \frac{16\mathbf{x}^3}{\mathbf{x}^4 - 81}\right]$$

$$= \frac{\left(\mathbf{x}^2 - 1\right)^2 \left(\mathbf{x}+2\right)^3}{\left(\mathbf{x}^4 - 81\right)^4}\left[\frac{4\mathbf{x}}{\mathbf{x}^2 - 1} + \frac{3}{\mathbf{x}+2} - \frac{16\mathbf{x}^3}{\mathbf{x}^4 - 81}\right]$$ (Leave in this form!)

