# Rates of Change

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Recall that there are two (2) types of rates of change:

1. **Average Rate of Change** with respect to an *interval* – Difference Quotient
2. **Instantaneous Rate of Change** with respect to a *point* – First Derivative

Presently, we are mostly considering instantaneous rates of change, that is, derivative related changes:

Given $\mathbf{f}(\mathbf{x})$, find

1. Average Rate of Change: $\dfrac{f(x+\Delta\mathbf{x}) - \mathbf{f}(\mathbf{x})}{\Delta\mathbf{x}}$
2. Instantaneous Rate of Change: $f'(x) = \lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta\mathbf{x}) - \mathbf{f}(\mathbf{x})}{\Delta\mathbf{x}}$

**Example 01:** Given $\mathbf{f}(\mathbf{t}) = \dfrac{2\mathbf{t}}{\mathbf{t}+2}$, find

1. Average velocity on $[2, 2.25]$
2. Instantaneous velocity at $\mathbf{t} = 2$

**Solution:**

$$\text{Average Velocity: } \frac{\mathbf{f}(\mathbf{t}_0 + \Delta\mathbf{t}) - \mathbf{f}(\mathbf{t}_0)}{\Delta\mathbf{t}} = \frac{\mathbf{f}(\mathbf{t}_1) - \mathbf{f}(\mathbf{t}_0)}{\mathbf{t}_1 - \mathbf{t}_0}$$

$$\text{where } [\mathbf{t}_0, \mathbf{t}_1]\ ;\ \Delta\mathbf{t} = \mathbf{t}_1 - \mathbf{t}_0$$

$$\mathbf{t}_0 = 2\ ;\ \mathbf{t}_1 = 2.25 \Rightarrow \frac{\mathbf{f}(\mathbf{t}_1) - \mathbf{f}(\mathbf{t}_0)}{\mathbf{t}_1 - \mathbf{t}_0} = \frac{\mathbf{f}(2.25) - \mathbf{f}(2)}{2.25 - 2}$$

$$= \dots = \frac{4}{17}\ \mathbf{ft}/\text{sec}$$

 

Instantaneous Velocity: $\mathbf{f}'(\mathbf{t})\big|_{\mathbf{t}=\mathbf{t}_0}$

$$\mathbf{f}'(\mathbf{t}) = 2\frac{(\mathbf{t}+2)*1-\mathbf{t}*1}{(\mathbf{t}+2)^2} = \frac{4}{(\mathbf{t}+2)^2}$$

$$\mathbf{f}'(2) = \mathbf{f}'(\mathbf{t})\big|_{\mathbf{t}=2} = \frac{2}{8} = \frac{1}{4}\ \mathbf{ft}/\sec$$

**Example 02:** Given $\mathbf{T}(\mathbf{t}) = 6 + 3\mathbf{t} + \frac{1}{\mathbf{t}+2} \quad \mathbf{t} \in [0,10]$ where $\mathbf{T}(\mathbf{t})$ in degrees $C^\circ$ and t in minutes, find $\mathbf{T}'(\mathbf{t})$ at t = 1.

**Solution:**

Calculate Generic Derivative:

$$\mathbf{T}(\mathbf{t}) = 6 + 3\mathbf{t} + (\mathbf{t}+2)^{-1}$$

$$\Rightarrow \mathbf{T}'(\mathbf{t}) = 3 - (\mathbf{t}+2)^{-2} = 3 - \frac{1}{(\mathbf{t}+2)^2}$$

Evaluate Generic Derivative:

$$\mathbf{T}'(1) = 3 - \frac{1}{(1+2)^2} = 3 - \frac{1}{9} = 2\frac{8}{9}\ \frac{\mathbf{C}^\circ}{\min}$$

 

**Example 3: Boyle's Law for Gases**

Boyle's Law: $\mathbf{PV}=\mathbf{C}=\text{constant}\quad \mathbf{P}(\mathbf{t})=\text{pressure}\ ;\ \mathbf{V}(\mathbf{t})=\textbf{volume}$

Note: $\mathbf{P}(\mathbf{t})\mathbf{V}(\mathbf{t})=\mathbf{C}$ ; $\mathbf{C}$ depends upon the gas under consideration

Note: There must always be enough data present to find $\mathbf{C}$:Given $\mathbf{t}_0 \Rightarrow$

$$\mathbf{C}=\mathbf{P}(\mathbf{t}_0)\mathbf{V}(\mathbf{t}_0)$$

Find $\mathbf{V}'(\mathbf{t})\Big|_{\mathbf{t}=2.5}$ when $\mathbf{P}(\mathbf{t})=10+4\mathbf{t}\ \dfrac{\mathbf{cm}}{\mathbf{Hg}}\quad \mathbf{t}\in[0,10]$ with

$\mathbf{V}(0)=20\ \text{cm}^3$and $\mathbf{P}(0)=10$

**Solution:**

We have $\mathbf{C}=\mathbf{V}(0)\mathbf{P}(0)=20*10=300\ \dfrac{\mathbf{cm}^4}{\mathbf{Hg}}$ so that

$$\mathbf{V}(\mathbf{t})=\frac{300}{10+4\mathbf{t}}\ \mathbf{cm}^3=300(10+4\mathbf{t})^{-1}\ \mathbf{cm}^3\quad \mathbf{t}\in[0,10]$$

$$\mathbf{V}'(\mathbf{t})=-300(10+4\mathbf{t})^{-2}(4)=-\frac{1200}{(10+4\mathbf{t})^2}\ \frac{\mathbf{cm}^3}{\text{min}}$$

$$\mathbf{V}'(2.5)=-\frac{1200}{(10+10)^2}=-\frac{1200}{400}=-3\ \frac{\mathbf{cm}^3}{\text{min}}$$

 

**Definition:** Let $\mathbf{s}(\mathbf{t})$ represent the position of a particle at time $\mathbf{t}$.
The velocity of the particle is given by $\mathbf{v}(\mathbf{t}) = \mathbf{s}'(\mathbf{t})$.
**Note:** $\mathbf{v}(\mathbf{t}) > 0 \Rightarrow$ motion upward ; $\mathbf{v}(\mathbf{t}) < 0 \Rightarrow$ motion downward
The acceleration of a particle is given by $\mathbf{a}(\mathbf{t}) = \mathbf{v}'(\mathbf{t}) = \mathbf{s}''(\mathbf{t})$

**Example 04:** A particle moving vertically has a position defined by

$$\mathbf{s}(\mathbf{t}) = 160\mathbf{t} - 16\mathbf{t}^2 \ \mathbf{ft}$$

Find its velocity, acceleration, the maximum height it will reach, and its velocity when it hits the ground.

**Solution:**

We have

$$\mathbf{v}(\mathbf{t}) = \mathbf{s}'(\mathbf{t}) = 160 - 32\mathbf{t} \ \mathbf{ft}/\text{sec}$$

$$\mathbf{a}(\mathbf{t}) = \mathbf{v}'(\mathbf{t}) = \mathbf{s}''(\mathbf{t}) = -32 \ \mathbf{ft}/\text{sec}^2$$

Note the $\mathbf{v}(3) = +54$ so the motion is upward and $\mathbf{v}(7) = -54$ so the motion is downward. The velocity is zero at its maximum height:

$$0 = \mathbf{v}(\mathbf{t}) = \mathbf{s}'(\mathbf{t}) = 160 - 32\mathbf{t} \ \mathbf{ft}/\text{sec} \Rightarrow \mathbf{t} = 5\text{sec}$$

Thus, the maximum height is $\mathbf{s}(5) = 160(5) - 16(5)^2 = 400 \ \mathbf{ft}$.
At the ground $0 = \mathbf{s}(\mathbf{t}) = 160\mathbf{t} - 16\mathbf{t}^2 \ \mathbf{ft} \Rightarrow \mathbf{t} = 10 \text{ sec}$ so that
$\mathbf{v}(10) = -160 \ \mathbf{ft}/\text{sec}$

 