Related Rates of Change

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If two variables are related by some equation and we know the rate of change (first derivative) of one of them, we can use implicit differentiation to find the rate of change of the other variable since they are "related".

Note: This idea can be extended to equations containing 3 or more variables.

Note: Frequently, we first must construct the defining equation from the problem itself.

Consider the area of a circle:

$$\mathbf{A} = \pi \mathbf{r}^2$$

If the radius r is changing and we know how, shouldn't this formula help us to find how the area A is changing? The answer is yes and is found using implicit differentiation. If the radius and therefore the area is a function of time t, we can use the formula above to obtain a formula relating their rates:

$$\mathbf{A}(\mathbf{t}) = \pi\ [\mathbf{r}(\mathbf{t})]^2 \Rightarrow \frac{\mathbf{dA}}{\mathbf{dt}} = \pi\left\{2\ [\mathbf{r}(\mathbf{t})]\ \frac{\mathbf{dr}}{\mathbf{dt}}\right\}$$

If, for example, $\mathbf{r}(\mathbf{t}) = 3\,\mathbf{in}\ \&\ \frac{\mathbf{dr}}{\mathbf{dt}} = \frac{1}{4}\mathbf{in}/\text{sec}$ then $\frac{\mathbf{dA}}{\mathbf{dt}} = \pi\left\{2[3]\frac{1}{4}\right\} = \frac{3\pi}{2}\mathbf{in}/\text{sec}$

Example 01: Given $\mathbf{y} = \sqrt{\mathbf{x}}$. If $\frac{\mathbf{dx}}{\mathbf{dt}} = 6$ when $\mathbf{x} = 9$, find $\frac{\mathbf{dy}}{\mathbf{dt}}$.

Solution: We have

$$\frac{\mathbf{dy}}{\mathbf{dt}} = \frac{1}{2}\mathbf{x}^{-1/2}\frac{\mathbf{dx}}{\mathbf{dt}} = \frac{1}{2\sqrt{\mathbf{x}}}\frac{\mathbf{dx}}{\mathbf{dt}} \Rightarrow \frac{\mathbf{dy}}{\mathbf{dt}} = \frac{1}{2\sqrt{9}}6 = 1$$

Example 02: Given $\mathbf{y} = \mathbf{x}\ \mathbf{e}^{-\mathbf{x}}$. If $\dfrac{\mathbf{dy}}{\mathbf{dt}} = 3$ when $\mathbf{x} = 0$, find $\dfrac{\mathbf{dx}}{\mathbf{dt}}$.

Solution:

We have

$$\frac{\mathbf{dy}}{\mathbf{dt}} = \mathbf{x}\left[-\mathbf{e}^{-\mathbf{x}}\frac{\mathbf{dx}}{\mathbf{dt}}\right] + \mathbf{e}^{-\mathbf{x}}\frac{\mathbf{dx}}{\mathbf{dt}}$$

$$= \mathbf{e}^{-\mathbf{x}}\left(1-\mathbf{x}\right)\frac{\mathbf{dx}}{\mathbf{dt}}$$

$$\Rightarrow 3 = 1(1)\frac{\mathbf{dx}}{\mathbf{dt}} \Rightarrow \frac{\mathbf{dx}}{\mathbf{dt}} = 3$$

Example 03: Given $\mathbf{xy}^2 - \mathbf{yx}^2 = 4$. If $\dfrac{\mathbf{dx}}{\mathbf{dt}} = 4$ when $\left(\mathbf{x},\mathbf{y}\right) = \left(2,-1\right)$, find $\dfrac{\mathbf{dy}}{\mathbf{dt}}$.

Solution: We have

$$\mathbf{x}\left[2\mathbf{y}\frac{\mathbf{dy}}{\mathbf{dt}}\right] + \mathbf{y}^2\left[\frac{\mathbf{dx}}{\mathbf{dt}}\right] - \mathbf{y}\left[2\mathbf{x}\frac{\mathbf{dx}}{\mathbf{dt}}\right] - \mathbf{x}^2\left[\frac{\mathbf{dy}}{\mathbf{dt}}\right] = 0$$

$$\Rightarrow 2\left(-2\frac{\mathbf{dy}}{\mathbf{dt}}\right) + 1(4) - (-1)(4*4) - 4\frac{\mathbf{dy}}{\mathbf{dt}} = 0$$

$$\Rightarrow 8\frac{\mathbf{dy}}{\mathbf{dt}} = 20 \Rightarrow \frac{\mathbf{dy}}{\mathbf{dt}} = \frac{5}{2}$$

Example 04: If the radius r of a sphere is changing at the rate of 2 ft/min when r = 5 ft, what rate is the volume changing?

Solution: The formula relating the volume to the radius is given by

$$\mathbf{V}(\mathbf{t}) = \frac{4\pi}{3}\left[\mathbf{r}(\mathbf{t})\right]^3$$

Thus

$$\frac{\mathbf{dV}}{\mathbf{dt}} = \frac{4\pi}{3}\left\{3\left[\mathbf{r}(\mathbf{t})\right]^2\frac{\mathbf{dr}}{\mathbf{dt}}\right\} \Rightarrow \frac{\mathbf{dV}}{\mathbf{dt}} = 4\pi\left\{5^2 * 2\right\}$$

$$= 200\pi\,\mathbf{cubic\,ft}\,/\,\mathrm{min}$$