Equations – Absolute Value

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Your Personal Mathematics Trainer
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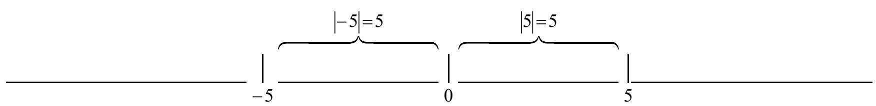
Definition: The **absolute value** of a number "x", written $|\mathbf{x}|$, refers to the size (magnitude, distance from 0, …). It will always be either 0 ($|0| = 0$) or a positive number ($|-5| = 5 = |5|$).



Note: $|\mathbf{x}| \geq 0$ & $|\mathbf{x}| \not< 0$

INFORMALLY: If x < 0, to find the absolute value of x, just through away the minus sign: $|-5| = 5$. If x > 0, the absolute value of x is just x: $|5| = 5$.

FORMALY:

$$|\mathbf{x}| = \begin{cases} -\mathbf{x} \text{ if } \mathbf{x} < 0 \\ 0 \text{ if } \mathbf{x} = 0 \\ \mathbf{x} \text{ if } \mathbf{x} > 0 \end{cases}$$

Using the "formal definition", we have

$$|-5| = -(-5) = 5$$
$$|0| = 0$$
$$|5| = 5$$

We obtain the same results if we do things "informally". For now, just throw away the minus sign if there is one: $|-5| = 5$

KEY: The solution of $|\mathbf{u}| = \mathbf{b}\ ;\ \mathbf{b} \geq 0$ is the ***union*** of the solutions of

a. $\mathbf{u} = -\ \mathbf{b}$

b. $\mathbf{u} = +\ \mathbf{b}$

TRADE $|\mathbf{u}| = \mathbf{b}\ ;\ \mathbf{b} \geq 0$ **for** $\mathbf{u} = -\ \mathbf{b}$ **AND** $\mathbf{u} = +\ \mathbf{b}$

Note: An absolute value equation trades for two (2) other equations!

Question 01: Solve for x: $|5-6\mathbf{x}|=3$

Solution:

Step	Equation		Reason
0	$\lvert 5-6\mathbf{x}\rvert=3$		
1	$5-6\mathbf{x}=-3$ $6\mathbf{x}=8$ $\mathbf{x}=\frac{8}{6}=\frac{4}{3}$	$5-6\mathbf{x}=3$ $6\mathbf{x}=2$ $\mathbf{x}=\frac{1}{3}$	Trade for 2 Equations

Graph of the solution set:

$$\underset{0}{|}\quad \underset{|}{1/3}\quad \underset{|}{4/3}$$

Question 02: Solve for x: $|\mathbf{x}^2+2\mathbf{x}|=2$

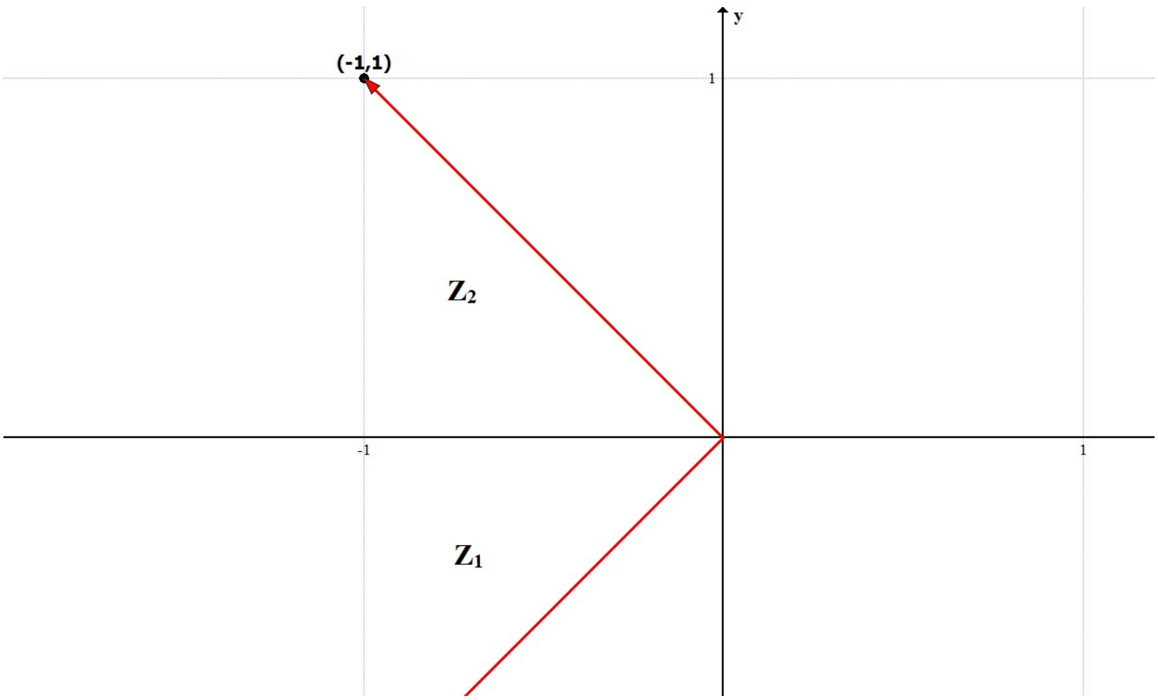
Solution:

Step	Equation		Reason
0	$\lvert\mathbf{x}^2+2\mathbf{x}\rvert=2$		
1	$\mathbf{x}^2+2\mathbf{x}=-2$ $\mathbf{x}^2+2\mathbf{x}+2=0$ $\mathbf{x}=\frac{-2\pm\sqrt{2^2-4(1)(2)}}{2}$ $=\frac{-2\pm\sqrt{-4}}{2}$ $=\frac{-2\pm 2\mathbf{i}}{2}$ $=-1\pm\mathbf{i}$	$\mathbf{x}^2+2\mathbf{x}=2$ $\mathbf{x}^2+2\mathbf{x}-2=0$ $\mathbf{x}=\frac{2\pm\sqrt{2^2-4(1)(-2)}}{2}$ $\mathbf{x}=\frac{2\pm\sqrt{12}}{2}$ $=\frac{2\pm 2\sqrt{3}}{2}$ $=1\pm\sqrt{3}$	Trade for 2 Equations

Graphs: Four (4) Solutions

$$\_\_\_\_\_\_\_\_ 1-\sqrt{3} \_\_\_\_\_\_\_\_\_ \underset{0}{|} \_\_\_\_\_\_\_\_\_\_\_\_ 1+\sqrt{3} \_\_\_\_\_\_\_$$

AND



FYI 01: Consider $\mathbf{x}^2 + 2\mathbf{x} + 2 = 0$

$$\begin{aligned}
&\mathbf{x} = -1 - \mathbf{i} \ \& \ \mathbf{x} = -1 + \mathbf{i} \\
&\mathbf{x} + 1 + \mathbf{i} = 0 \ \& \ \mathbf{x} + 1 - \mathbf{i} = 0 \\
&(\mathbf{x} + 1 + \mathbf{i})(\mathbf{x} + 1 - \mathbf{i}) = 0 \\
&\mathbf{x}^2 + \mathbf{x} - \mathbf{xi} + \mathbf{x} + 1 - \mathbf{i} + \mathbf{xi} + \mathbf{i} - \mathbf{i}^2 = 0 \\
&\mathbf{x}^2 + 2\mathbf{x} + 2 = 0
\end{aligned}$$

Hence $\mathbf{x}^2 + 2\mathbf{x} + 2$ factors as $(\mathbf{x} + 1 + \mathbf{i})(\mathbf{x} + 1 - \mathbf{i})$ **UGLY Factors!**

Substituting $\mathbf{x} = -1 + \mathbf{i}$ into $\mathbf{x}^2 + 2\mathbf{x} + 2$ yields

$$\begin{aligned}
\mathbf{x}^2 + 2\mathbf{x} + 2 &= (-1 + \mathbf{i})^2 + 2(-1 + \mathbf{i}) + 2 \\
&= 1 - 2\mathbf{i} + \mathbf{i}^2 - 2 + 2\mathbf{i} + 2 \\
&= 0
\end{aligned}$$

This calculation verifies that $\mathbf{x} = -1 + \mathbf{i}$ is a solution of $\mathbf{x}^2 + 2\mathbf{x} + 2 = 0$

Your Turn: Show that $\mathbf{x} = -1 - \mathbf{i}$ is also a solution of $\mathbf{x}^2 + 2\mathbf{x} + 2 = 0$

FYI 02: Consider $\mathbf{x}^2 + 2\mathbf{x} - 2 = 0$

$$\mathbf{x} = -1 - \sqrt{3} \text{ \& } \mathbf{x} = -1 + \sqrt{3}$$
$$\mathbf{x} + 1 + \sqrt{3} = 0 \text{ \& } \mathbf{x} + 1 - \sqrt{3} = 0$$
$$\left(\mathbf{x} + 1 + \sqrt{3}\right)\left(\mathbf{x} + 1 - \sqrt{3}\right) = 0$$
$$\mathbf{x}^2 + \mathbf{x} - \sqrt{3}\ \mathbf{x} + \mathbf{x} + 1 - \sqrt{3} + \sqrt{3}\ \mathbf{x} + \sqrt{3} - \left(\sqrt{3}\right)^2 = 0$$
$$\mathbf{x}^2 + 2\mathbf{x} - 2 = 0$$

Hence $\mathbf{x}^2 + 2\mathbf{x} + 2$ factors as $\left(\mathbf{x} + 1 + \sqrt{3}\right)\left(\mathbf{x} + 1 - \sqrt{3}\right)$ **UGLY Factors!**

Substituting $\mathbf{x} = -1 + \sqrt{3}$ into $\mathbf{x}^2 + 2\mathbf{x} + 2$ yields

$$\begin{aligned}\mathbf{x}^2 + 2\mathbf{x} - 2 &= \left(-1 + \sqrt{3}\right)^2 + 2\left(-1 + \sqrt{3}\right) - 2 \\ &= 1 - 2\sqrt{3} + \left(\sqrt{3}\right)^2 - 2 + 2\sqrt{3} - 2 \\ &= 0\end{aligned}$$

This calculation verifies that $\mathbf{x} = -1 + \sqrt{3}$ is a solution of $\mathbf{x}^2 + 2\mathbf{x} - 2 = 0$

Your Turn: Show that $\mathbf{x} = -1 - \sqrt{3}$ is also a solution of $\mathbf{x}^2 + 2\mathbf{x} - 2 = 0$

