FUNctions – Domain

[Allowable Inputs: x-values]
[Projection of graph onto the x-axis]

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Basic Function Idea/Concept – For each “allowable input:”, there corresponds a “unique output”:

Allowable Input $\overset{\textbf{implies}}{\Rightarrow}$ **Unique Output**

A menu is a “simple” function:

Menu Items	**Price**
Fries	**$1.95**
Cheeseburger	**$3.75**
Steak Sandwich	**$4.50**
Iced Tea	**$150**
Lemonade	**$1.50**

The list of menu items is the domain. Each item corresponds to a unique price:

Cheeseburger → **$3.75**

Note that different items on the menu are allowed to correspond to the same price:

Iced Tea → $1.50

Lemonade → $1.50

The functions **f** we consider involve just real numbers and are referred to as "**real-valued functions**". That is, both "x" & "y = f(x)" must be real numbers.

Assumption: We consider **s**, that is, functions that take real #'s to real #'s

Recall from our set theory material:

1. The symbol $"\subseteq"$ means "contained in" or "a subset of"
2. The symbol $"\in"$ means "is an element of" "or "member of"
3. $\mathbb{R}_x$ represents a copy of the *horizontal* number line
4. $\mathbb{R}_y$ represents a copy of the *vertical* number line

Using these symbols, we present the technical definition of a function:

Definition: A **function** $\mathbf{f}$ from a set $\mathbf{X} \subseteq \mathbb{R}_x$ unto a set $\mathbf{Y} \subseteq \mathbb{R}_y$ is a correspondence that associates with each $\mathbf{x} \in \mathbf{X}$ one and only one $\mathbf{y} \in \mathbf{Y}$. This is sometimes denoted **f : X → Y.** Also, the correspondence, rule, formula, … is frequently denoted by **f(x): y = f(x).**

$\mathbf{X} \subseteq \mathbb{R}_x$ is called the **domain**: $\mathbf{Domain\ f = Dom\ f}$

Note: The *domain* of $\mathbf{f}$ is a subset of all the real numbers on the *horizontal* number line.

$\mathbf{Y} \subseteq \mathbb{R}_y$ is called the **range**: $\mathbf{Range\ f = Rng\, f}$

Note: The *range* of **f** is a subset of all the real numbers on the *vertical* number line.

To *officially* specify a function, we need

1. Name
2. Symbol
3. Domain
4. Correspondence (rule, formula, …)
5. Range

Note, however, that all of this information is usually not initially given but can be determined.

The **domain** of **f** may be defined two (2) ways:

1. **Explicitly** – the domain is given

(1) Consider **f** defined by the following table:

x	**y = f(x)**
3	**7**
-2	**1**
4	**-3**
6	**2**

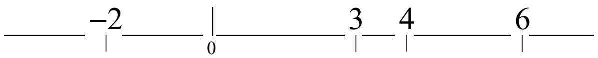
Its domain is the *vertical* "**x**" column: **Dom f** $= \{-2, 3, 4, 6\}$. By the way, **Range f** $= \{-3, 1, 2, 7\}$

Note: $\mathbf{f}(-2) = 1$

Note: $\mathbf{f}(4) = -3$

Note: $\mathbf{f}(7) =$ undefined (does NOT exist, TRASH, $\nexists$) since $\mathbf{x} = 7 \notin \mathbf{Dom\ f}$

Geometrically, **Dom f** is represented by



(2) Consider $\mathbf{y} = \mathbf{f}(\mathbf{x}) = 4\mathbf{x}^2 - 2\mathbf{x}$; $\mathbf{Dom\ f} = [-3, 2)$

We have

1. $\mathbf{y} = \mathbf{f}(-3) = 4(-3)^2 - 2(-3) = 42$
2. $\mathbf{y} = \mathbf{f}\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)^2 - 2\left(\frac{1}{2}\right) = 0$
3. $\mathbf{y} = \mathbf{f}(1) = 4(1)^2 - 2(1) = 2$
4. $\mathbf{y} = \mathbf{f}(2) = \nexists$; note that $\mathbf{x} = 2 \notin \mathbf{Dom\ f}$ due to the ")"

Remember ")" means not included and "[" means included

Note: If the domain is given, we are *mandated* to use it!

2. **Implicitly** – the domain is defined by $\mathbf{Dom\ f} = \{\mathbf{x} \in \mathbb{R}_{\mathbf{x}} \mid \mathbf{f}(\mathbf{x}) \in \mathbb{R}_{\mathbf{y}}\}$, so we must determine it ourselves. Recall that "{}" refers to a set of elements, in this case real numbers, and "|" means "such that". We must determine all the real numbers **x** such that **y = f(x)** is also a real number. It is *usually* easier to determine which real numbers must be rejected.

There are two (2) ways of excluding a real number $\mathbf{x}$ from being in the domain of a function $\mathbf{f}$:

1. **Division by zero**: A value for $\mathbf{x}$ can NEVER make the denominator zero!

$$\frac{\text{Numerator}}{\text{Denominator} \neq 0}$$

2. **Negative under an even root**: A value for $\mathbf{x}$ can NEVER make an expression under an even root negative!

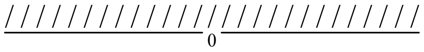
$$\sqrt[\text{EVEN}]{\text{Expression} \nless 0}$$

The second condition disallows "complex numbers".

Note: We will represent the domains both analytically and geometrically.

(3) Find the domain of $\mathbf{f}(\mathbf{x}) = -3$.

Solution: The domain is given by $\textbf{Dom f} = \mathbb{R}_{\mathbf{x}} = (-\infty, +\infty)_{\mathbf{x}}$ since there is no way to divide by zero and there are no even roots in the formula defining $\mathbf{f}$:



(4) Find the domain of $\mathbf{f}(\mathbf{x}) = 3 - 2\mathbf{x}$.

Solution: The domain is given by $\textbf{Dom f} = \mathbb{R}_{\mathbf{x}} = (-\infty, +\infty)_{\mathbf{x}}$ since there is no way to divide by zero and there are no even roots in the formula defining $\mathbf{f}$:

$$\underline{/////////////}_{0}\underline{/////////////}$$

(5) Find the domain of $\mathbf{f}(\mathbf{x})=15\mathbf{x}^2-2\mathbf{x}-8$.

Solution: The domain is given by $\mathbf{Dom\ f}=\mathbb{R}_{\mathbf{x}}=(-\infty,+\infty)_{\mathbf{x}}$ since there is no way to divide by zero and there are no even roots in the formula defining $\mathbf{f}$:

$$\underline{/////////////}_{0}\underline{/////////////}$$

Note: The functions above are examples of **polynomial functions** and polynomial functions always satisfy:

$$\mathbf{Dom\ Polynomial}=\mathbb{R}_{\mathbf{x}}$$

(6) Find the domain of $\mathbf{f}(\mathbf{x})=\dfrac{\mathbf{x}}{\mathbf{x}-2}$.

Solution: Since division by zero is NOT allowed, $\mathbf{x}-2\neq 0$ so that $\mathbf{x}\neq 2$.
The domain is thus given by $\mathbf{Dom\ f}=\mathbb{R}_{\mathbf{x}}\setminus\{2\}=(-\infty,2)\cup(2,+\infty)_{\mathbf{x}}$:

$$\underline{/////////////}_{0}\underline{/////}\times_{2}\underline{//////////}$$

(7) Find the domain of $\mathbf{f}(\mathbf{x})=\dfrac{2\mathbf{x}+5}{3\mathbf{x}^2-2\mathbf{x}-8}$.

Solution: Division by zero is NOT allowed, that is, $3\mathbf{x}^2-2\mathbf{x}-8\neq 0$:

Step	Equation	Reason
0	$3\mathbf{x}^2-2\mathbf{x}-8=0$	Solutions will NOT be in the domain
1	$(3\mathbf{x}+4)(\mathbf{x}-2)=0$	Factor

2	$3\mathbf{x}+4=0$ $\mathbf{x}=-\frac{4}{3}$	$\mathbf{x}-2=0$ $\mathbf{x}=2$	Trading a quadratic equation for two linear equations

The domain is given by $\mathbf{Dom\ f}=\mathbb{R}_{\mathbf{x}}\setminus\left\{-\frac{4}{3},2\right\}=\left(-\infty,-\frac{4}{3}\right)\cup\left(-\frac{4}{3},2\right)\cup\left(2,+\infty\right)_{\mathbf{x}}$:

$\underline{//////////////}\underset{-4/3}{\times}\underline{///}_{0}\underline{//////}\underset{2}{\times}\underline{//////////}$

(8) Find the domain of $\mathbf{f}(\mathbf{x})=\sqrt{2-5\mathbf{x}}$.

Solution: An expression under an **even root** cannot be negative: $2-5\mathbf{x}\nless 0$:

Step	Inequality	Reason
0	$2-5\mathbf{x}\geq 0$	Solutions are domain elements
1	$2\geq 5\mathbf{x}$	
2	$\frac{2}{5}\geq\mathbf{x}$ OR $\mathbf{x}\leq\frac{2}{5}$	

Thus the domain is given by $\mathbf{Dom\ f}=\left(-\infty,\frac{2}{5}\right]_{\mathbf{x}}$: $\underline{/////////////////}_{0}\underline{/////}\ ]_{2/5}\underline{\qquad\qquad}$

(9) Find the domain of $\mathbf{f}(\mathbf{x})=\sqrt{12-\mathbf{x}-\mathbf{x}^2}$.

Solution: An expression under an *even* root cannot be negative: $12-\mathbf{x}-\mathbf{x}^2 \nless 0$:

Step	Inequality	Reason
0	$12-\mathbf{x}-\mathbf{x}^2 \ge 0$	Domain elements
1	$(3-\mathbf{x})(4+\mathbf{x}) \ge 0$	Factor
2	Determine Boundary Points: $(3-\mathbf{x})(4+\mathbf{x}) = 0$ $\mathbf{x}=-4$; $\mathbf{x}=3$	
3	Check Boundary Points: 1. $\mathbf{x}=-4$: Yes! 2. $\mathbf{x}=3$: Yes!	
4	Check Intervals: 1. $(-\infty,-4)$: Test Point $\mathbf{x}=-10$; $12-[-10]-[-10]^2 \overset{?}{\ge} 0$ False 2. $(-4,3)$: Test Point $\mathbf{x}=0$; $12-[0]-[0]^2 \overset{?}{\ge} 0$ True 3. $(3,+\infty)$: Test Point $\mathbf{x}=10$; $12-[10]-[10]^2 \overset{?}{\ge} 0$ False	
5	Solution: $[-4,3]$	

The domain is given by **Dom f** $=[-4,3]$:

$\underline{\qquad\qquad}\underset{-4}{[}\,\underline{/////}\underset{0}{}\underline{/////}\,\underset{3}{]}\underline{\qquad\qquad}$

(10) Find the domain of $\mathbf{f}(\mathbf{x}) = \dfrac{\sqrt{\mathbf{x}+4}}{\mathbf{x}-6}$.

Solution: An expression under an even root cannot be negative: $\mathbf{x}+4 \nless 0$:

Step	Inequality	Reason
0	$\mathbf{x}+4 \geq 0$	Possible Domain elements
1	$\mathbf{x} \geq -4$	

Also, the denominator cannot be zero: $\mathbf{x}-6 \neq 0$ so that $\mathbf{x} \neq 6$. Hence, the domain is given by

$\mathbf{Dom\ f} = [-4,6) \cup (6,+\infty)_{\mathbf{x}}$: $\underline{\qquad\qquad}\underset{-4}{[}\,\underline{////////}\underset{0}{/}\underline{////////////}\underset{6}{\times}\underline{/////////////}$

(11) Find the domain of $\mathbf{f}(\mathbf{x}) = \begin{cases} 1+2\mathbf{x} & ; -3 \le \mathbf{x} < 2 \\ 1-2\mathbf{x} & ; \quad 2 \le \mathbf{x} < 5 \end{cases}$.

This is called a **piece-wise function**. We have

$\mathbf{f}(-7) = \text{Undefined}$

$\mathbf{f}(1) = 1 + 2(1) = 3$

$\mathbf{f}(3) = 1 - 2(3) = -5$

$\mathbf{f}(8) = \text{Undefined}$

Solution: The function $\mathbf{f}$ can be illustrated as follows:

$$\overbrace{\left[-3,2\right)}^{1-2\mathbf{x}}\overbrace{\left[2,5\right)}^{1+2\mathbf{x}}$$

The domain is the union of the two (2) sets of $\mathbf{x}$ values:

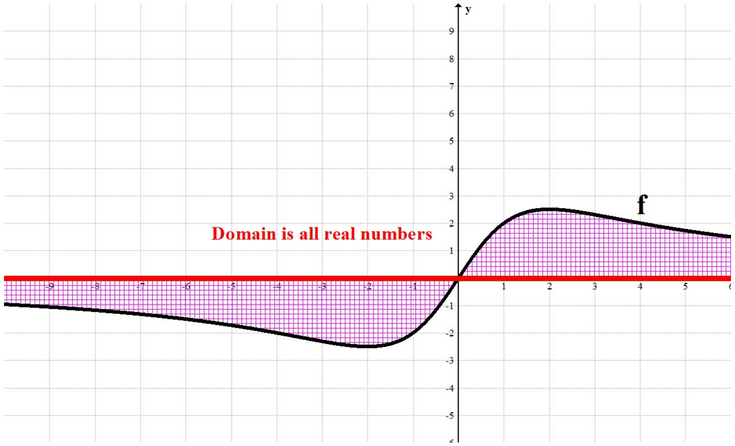
$$\textbf{Dom f} = \left[-3,2\right) \cup \left[2,5\right) = \left[-3,5\right).$$

Geometrically,

$$\underline{\qquad\qquad}\underset{-3}{[}\,\underline{///////}\underset{0}{/}\underline{///////////}\underset{5}{)}\underline{\qquad\qquad}$$

If the graph of the function $\mathbf{f}$ is known, the domain is the projection of the graph onto the x-axis:

FUNction 01:



FUNction 02: