FUNctions: Behavior at/toward Infinity

[What the f(x) values are doing when |x| is "big"]

$$\mathbf{f}(\mathbf{x}) \to ? \qquad \text{when} \qquad |\mathbf{x}| \to +\infty$$

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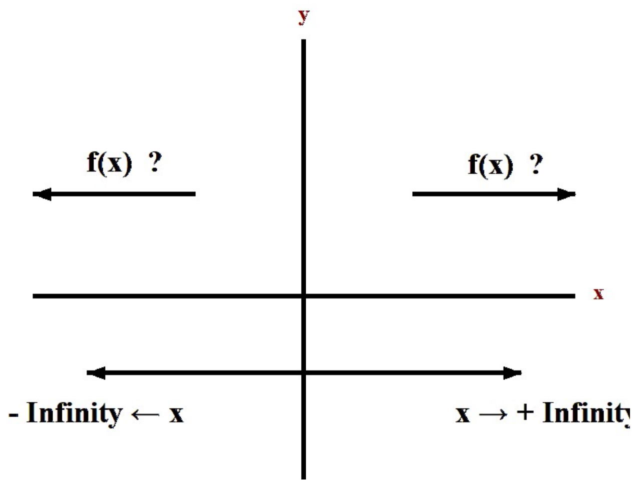
For the sake of discussion, let us assume that the domain of the function f under consideration is all the real numbers: $\mathbb{R}_{\mathbf{x}}$. To get a rough approximation to the graph of f, we "plot" some points on its graph including the intercept points and then connect them like we did with our "dot-to-dot" colorings book when we were young, assuming that there are no breaks in the graph. However, since we can only "plot" a finite number of points, our graphical representation will be lacking. To improve our representation, we determine if the f(x) values have a pattern as the x-values increase (decrease) without bound:



In symbols, the question is when $|\mathbf{x}| \to +\infty \overset{?}{\Rightarrow} \mathbf{y} = \mathbf{f}(\mathbf{x}) \to ?$

We concentrate on two (2) patterns the f(x) values can have:

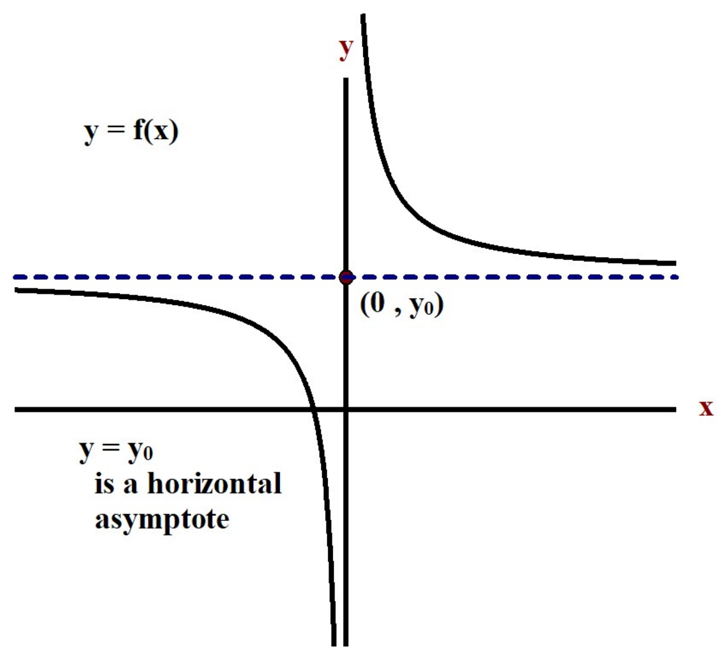
1. $|\mathbf{x}| \to +\infty \Rightarrow \mathbf{f}(\mathbf{x}) \to \pm\infty$
   As the x-values increase (decrease) without bound, the corresponding f(x) values increase (decrease) without bound.

2. $|\mathbf{x}| \to +\infty \Rightarrow \mathbf{f}(\mathbf{x}) \to \mathbf{y}_0 \in \mathbb{R}_{\mathbf{y}}$
   As the x-values increase (decrease) without bound, the corresponding f(x) values approach a number $\mathbf{y}_0$.

In the second case, we obtain what is called a **horizontal asymptote** of **f**: $\mathbf{y} = \mathbf{y}_0$:

Definition: A horizontal line $\mathbf{y} = \mathbf{y}_0$ is a **horizonal asymptote** of **f** if as the x-values increase (decrease) without bound (we say x "goes to $+\infty$ (or $-\infty$)") , the corresponding (x, f(x)) points get "closer and closer" to the line $\mathbf{y} = \mathbf{y}_0$.

Note: In other words, the graph of **f** gets "closer and closer" to the *horizontal* line $\mathbf{y} = \mathbf{y}_0$ as the x-values approach $+\infty$ (or $-\infty$).



Key Facts:

1. A horizontal asymptote $\mathbf{y} = \mathbf{y}_0$ may or may not intercept the graph.

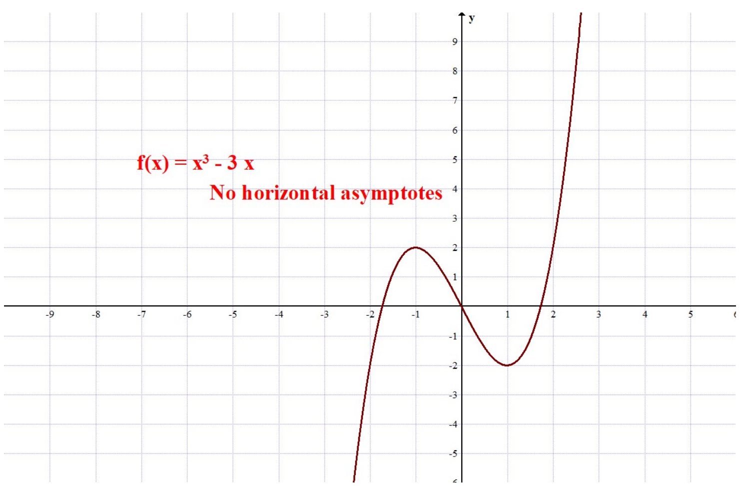
2. There can be 0, 1, or 2 horizontal asymptotes.

We are *just* considering the graphs below to identify the behavior of y = **f**(x) as $|\mathbf{x}| \to +\infty$. In future notes, we will show how to determine this behavior when we only have the formula for y = f(x).

Example 01: $\mathbf{y} = \mathbf{f}(\mathbf{x}) = \mathbf{x}^3 - 3\mathbf{x}$
Analysis:

Consider the graph of f we see . There are no horizontal asymptotes. In fact,

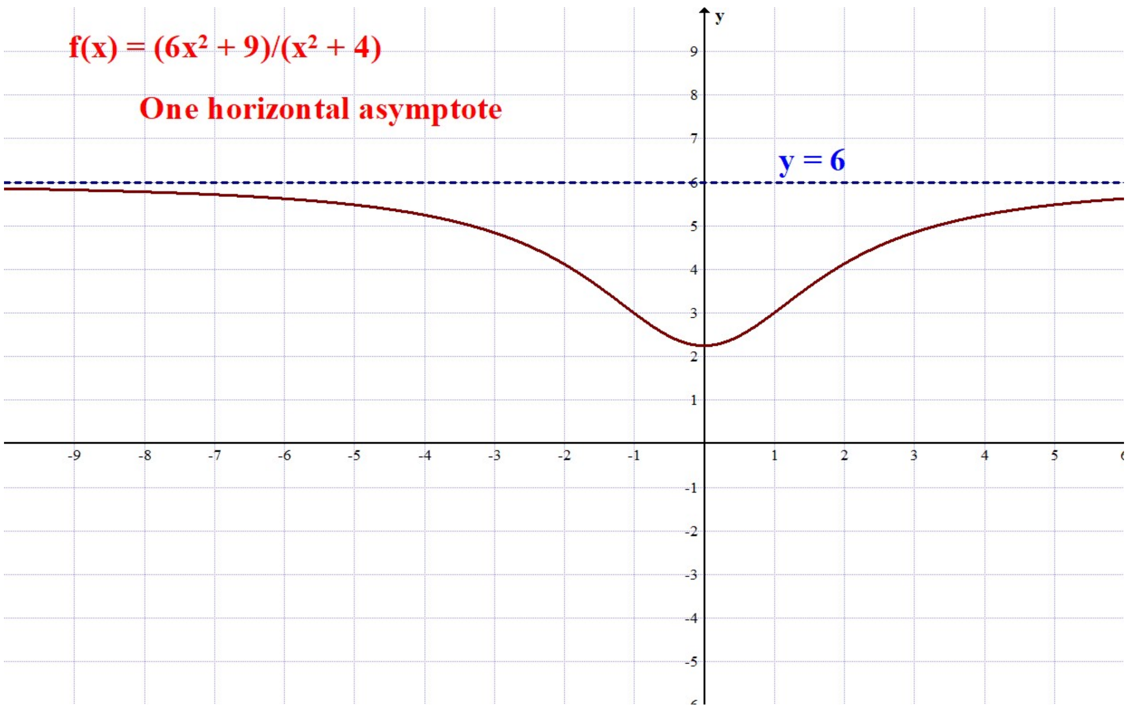
$$\mathbf{x} \to -\infty \Rightarrow \mathbf{f}(\mathbf{x}) \to -\infty \; ; \; \mathbf{x} \to +\infty \Rightarrow \mathbf{f}(\mathbf{x}) \to +\infty$$



Example 02: $y = f(x) = \dfrac{6x^2 + 9}{x^2 + 4}$

Analysis:

Considering the graph of f, we see $x \to -\infty \Rightarrow f(x) \to 6$; $x \to +\infty \Rightarrow f(x) \to 6$. Therefore y = 6 is a horizontal asymptote of f.



Example 03: $f(x) = \frac{\sqrt{2+4x^2}}{x}$

Analysis:

Considering the graph of **f**, we see $x \to -\infty \Rightarrow f(x) \to -2$; $x \to +\infty \Rightarrow f(x) \to +2$. Therefore **y = -2** *and* **y=2** are horizontal asymptotes of **f**.